Institute of Physics, University of Latvia
Riga Technical University

Vladislav Kremeneckis, Magister of Physics

Exact Self-Similar Solutions in Hydrodynamics and Magnetohydrodynamics and Their Relation to the Problems in the Boundary Layer Approximation

Thesis Synopsis
for the degree of Doctor of Physics
in the field of Mechanics of Fluids and Gases

Scientific Supervision:
E. Shcherbinin, Dr.Hab.Phys.
Institute of Physics, UL
(till August 2004)
J. Freibergs, Dr.Phys.
Institute of Physics, UL
(since August 2004)

Riga – Salaspils, January 2009

UDK 532:538.4

Kremeneckis V.
EXACT SELF-SIMILAR SOLUTIONS IN HYDRODYNAMICS AND
MAGNETOHYDRODYNAMICS AND THEIR RELATION
TO THE PROBLEMS IN THE BOUNDARY LAYER APPROXIMATION

Thesis Synopsis –R.:RTU, 2009. - 44 p.

ISBN

PROMOCIJAS DARBS
IZVIRZĪTS LATVIJAS UNIVERSITĀTĒ
FIZIKAS DOKTORA GRĀDA IEGŪŠANAI

Promocijas darbs fizikas doktora grāda iegūšanai tiek publiski aizstāvēts ___________________, ____

____.

OFICIĀLIE RECENZENTI:

____

____

____

____

APSTIPRINĀJUMS

Es apstiprinu, ka esmu izstrādājis doto promociju darbu, kas iesniegts izskatīšanai Latvijas Universitātē fizikas doktora grāda iegūšanai. Promocijas darbs nav iesniegts nevienā citā universitātē zinātniskā grāda iegūšanai.

Vladislavs Kremeņeckis......................................

Datums: __.______________ 2009. g.

SUMMARY

The thesis is concerned with the study of a possibility to find a common approach to self-similar solutions of exact stationary equations of hydrodynamics and magnetohydrodynamics and a relationship between the exact solutions and the boundary layer solutions.

The thesis considers four the most used systems of coordinates: a Cartesian system, a polar one (a particular case of the cylindrical coordinate system, which is usually considered as a self coordinate system), a cylindrical and a spherical systems (the latter two are considered as axially symmetric). The method of investigation is the search for stream functions allowing to transform the equations with partial derivatives into ordinary equations with the following derivation of equations themselves. Boundary conditions are not considered as they do not affect the equation derivation. For the magnetohydrodynamic flows, full equations are considered, where no suppositions about electromagnetic magnitudes are given, as well as equations in the non-inductive approximation, when induced magnetic fields are ignored.

As examples, some modifications of classical problems of hydrodynamics and magnetohydrodynamics are considered.

The thesis is written in Russian and consists of Introduction, 5 Sections and Conclusion. The total number of pages is 182 including 30 figures and 73 references.

This dissertation is written under the partial support of LZP and ESF.

CONTENTS

INTRODUCTION

Practical importance of the work. This work is concerned with systematization of the existing self-similar solutions of the equations of motion of a viscous incompressible fluid in ordinary hydrodynamics and in magnetohydrodynamics. Using a common approach in deriving self-similar solutions and the method of variable separation, all possible transformations of the hydrodynamic and magnetohydrodynamic equations in partial variables into ordinary differential equations have been investigated. Appropriate types of electric and magnetic field in the non-inductive approximation have been studied.

As known, the Navier-Stokes equation has no common solution [54]. With certain initial and boundary conditions, it is possible to derive a solution of this equation, but most of solutions are approximate, which have been obtained either by a simplified flow model or by approximate methods, for example, by asymptotic decompositions. Each of such solutions has a restricted area of application determined usually by critical values of some problem parameters. As a rule, it is necessary to search for solutions for several ranges of parameter values. Therefore, it is very important to find exact solutions unlimited by either parameter values.

One of the ways to obtain exact solutions is the self- similar approach [61], implying that equations with partial derivatives have to be transformed into ordinary differential equations, moreover, the variable for the ordinary equations is accepted as a combination of real variables. The solution of the ordinary equations is simpler than that of the equations with partial variables, but the application of the self- similar approach is restricted. In other words, the scope of problems, which could be solved by such approach, is limited. But the self-similar solution, which to some extent is an approximation, allows at least to define main features and parameters of the flow. Besides, the self-similar solution allows also to apply the once obtained solution to all similar problems by simply varying the values of the necessary parameters. Another application of the self-similar solution is its use like some initial approximation when solve (also numerically) a complicated problem, which cannot be solved as a self-similar problem [45].

The above-described method is applicable not only in hydrodynamics, but in magnetohydrodynamics as well. It is frequently used in the boundary layer theory because the proposed by L. Prandtl [28] transition from the Navier-Stokes equations to the boundary layer equations changes the type of differential equation from elliptic to parabolic that makes easier to introduce the self-similar variable. It is also possible to derive a boundary layer solution if in the exact solution (i.e., in full Navier-Stokes equations, where none of the terms is ignored) some parameter, characterizing the flow, tends towards some critical value. From the point of view of the equation, the tending of the parameter towards a critical value means that some terms become infinitesimally small if compared to other terms. As a result, it is possible to derive a boundary layer equation [43].

One more way to transform an equation with partial derivatives into an ordinary differential equation is the method of separation of variables. This method has a restricted area of application, too, but like the self-similar method, it allows to analyze the flow in a first approximation and apply the obtained solution as zero iteration for further investigation.

The self-similar approach is widely used in hydrodynamics and magnetohydrodynamics. There is a large number of problems solved within the framework of the discussed approach. This can be illustrated by the following examples: overflow of a semi-infinite plate by a running-on uniform flow (Blasius problem) [4], the Schlichting jet [68], the Akatnov near-wall jet [36], the problem of the front critical point (flow leaking-in on a flat surface) [68], mixing of parallel uniform flows (Görtler problem) [10], the Couette flow between parallel plates [68], the flow in a flat confuser (Polhausen problem) [26], the Hagen-Poiseuille flow in a tube [68], stratified flows (the flows with one velocity component) [68]. These problems have been formulated and solved in the Cartesian system of coordinates. Many problems have been solved in other coordinate systems, too. For example, a flow in confuser/diffuser (the Hamel flow) was solved in the polar system of coordinate [11]; the rotation of a disc in an infinite volume of fluid [14], a flow between two rotating discs [44], [49], a flow between two rotating discs of arbitrary shape [42], a circular Yatseev-Squire jet [73], [32], [33], a Loitsyansky radial-slot jet [59], a swirled flow in a tube [35], [65], a flow between two rotating co-axial cylinders (a circular analogue of the Couette flow) [68], a flow between two cones [62] were solved in the cylindrical coordinate system; in the spherical coordinates: a Landau jet [56], a Yatseev-Squire jet [73], [32], [33], a flow in a funnel with a vortical thread [70], a flow with a linear source in a circular cone [70], a flow with a linear injection/suction [70]. This list is far from being complete, nevertheless, it illustrates the applications of the self-similar approach as to a solution dependent on the combination of variables as to a case with variable separation. The derivation of self-similar solutions as well as the study of classical problems of stability are of interest nowadays, too ([1], [7], [8], [22], [23], [24], [25], [45]).

Most of the above said problems were considered for magnetic field cases that points out the applicability of the discussed method to transition from the equations with partial derivatives to ordinary differential equations in magnetohydrodynamics as well (see, e.g., [20], [29], [30], [31], [38-42], [43], [67], [69], [70], [71], [72].)

It can be added that only a small part of the above problems was solved in exact formulations, they were mainly solved in the boundary layer approximation. The same is referred to magnetohydrodynamics. Only a few exact solutions have been derived. It would be useful to determine for which cases exact solutions can be obtained in different coordinate systems.

The common feature almost for all self-similar problems is the fact that the solution begins with the derivation of a self-similar variable and self-similar equations, though it is done independent on the already solved problems. Common is just the approach. Yet, as E.Shcherbinin showed in his works ([38], [39], [40], [69], [71], [72]), all problems of the self-similar boundary layer are described by the only equation (in the hydrodynamic case) determined by a small number of coefficients, characterizing all peculiarities of the problem. Hence, there is no sense to derive this equation again and again – it is enough to define the values of the equation coefficients determined by the problem formulation. The same can be referred to magnetohydrodynamics – only the number of equations increases. The same common approach allows to obtain some flow characteristics (usually integral such as flowrate, impulse, and so on) without solving the equation itself, i.e., no calculation of the velocity field.

Besides, the existence of common equations, relating the fluid flow and the magnetic and electric fields, allows to define the types of appropriate electric and magnetic fields, which can be applied at problem formulation. i.e., from the given distribution of electric and magnetic fields one can decide whether it is possible to solve this problem in a self-similar approach or by the method of separation of variables. On the other hand, while choosing a way to make a problem solved in the self-similar approximation magnetic, one has to choose from a set of fields, which keep on the problem self-similar and affect the conducting fluid flow.

Defended statements. The following statements are defended in the thesis.

1. In each of the coordinate systems in question it is possible to derive universal

equations or systems of equations by the method of separation of variables and/or by the self-similar approach, which describe all possible versions of self-similar problems both in the precise formulation and in the approximation of boundary layer. To derive equations, describing a specific problem, from the problem conditions it is necessary to define some coefficients in these equations (e.g., configurations of the flow zone, boundary conditions, external fields and so on).

2. In the non-inductive approximation, there are the well-defined types of electric and magnetic fields that maintain the problems self-similar.
3. The method of variable separation is applicable only in polar and spherical coordinate systems. It is impossible to derive a boundary layer equation in these coordinate systems. To derive such a solution, it is necessary to solve exact equations, but the formation of a boundary layer is provided by changing a typical parameter (or parameters) of the problem.

Scientific novelty of the work. So far, the possibility to derive self-similar solutions in the framework of the boundary layer theory for the cylindrical and spherical systems of coordinates has been completely enough analyzed [43]. The application of the common approach to the problems under discussion in the Cartesian and polar coordinate systems was begun in the monograph [72] and relevant publications ([38], [39], [40], [69], [70]). Within the framework of this work, it is attempted to consider all possible versions of self-similar solutions as in precise formulation as in the boundary layer approximation for the four most used systems of coordinates. The work does not consider separately a version of transformation of the exact solution in the Cartesian and cylindrical coordinate systems to the boundary layer solutions by tending the parameter towards infinity.

At the same time, the derivation of ordinary differential equations by the method of separation of variables was considered only for the polar and spherical systems of coordinates, and in an inexplicit form for the cylindrical coordinate system ([43] [71], [72]). This work investigates a possibility to apply this method as to exact solutions as in the boundary layer approximation.

The obtained results can be applied also to investigate convective flows and flows with impurities. Partially, it is shown in the book [72]; some materials of which are included in the thesis.

The methods and the obtained results can be used in different areas of hydrodynamics and magnetohydrodynamics such as the theory of electrovortical flows, electroslag melting, plasma control, etc., where the interaction of the moving liquid and electromagnetic forces is significant. For example, a circular analogue of the Couette flow in a magnetic field is investigated to explain (understand) the phenomena occurring in stars [29].

The thesis considers the problems of hydrodynamics and magnetohydrodynamics, illustrating the proposed method applications and the derivation of boundary layer solutions from exact equations. It also discusses the influence of suction and injection of the fluid on the flow pattern. Some results obtained previously by other authors have been specified; the stability of some magnetohydrodynamics flows determined by the interaction of magnetic field and suction/injection was investigated.

Publications and Conferences

Publications

1. Kremeņeckis V., Ščerbiņins E., *Robežslāņa teorija hidrodinamikā aksiāli simetriskām un virpuļu plūsmām*, 35. RTU zinātniskā un tehniskā konference 1994. g. 19.-22. aprīlī. Konferences materiāli. lpp. 34-37. Rīga, 1994.
2. Kremenetsky V., Shcherbinin E., *Boundary layers formation in case of free and near-the-*

wall fan jets at the electrically induced vortical flows, Mahyd'95. Abstracts. p. 27. Jurmala, Latvia, 1995.
3. Bartulis A., Kremenetsky V., Shcherbinin E. *Theory of axisymmetric boundary layer of type II.* Magnetohydrodynamics, vol. 32, No.3, p.266-273, 1996.
4. Shcherbinin E., Bartulis A., Kremenetsky V., *Theory of axisymmetric boundary layer in hydrodynamics and magnethydrodynamics*. Proc. 3rd International Conference on Transfer Phenomena in Magnetohydrodynamics and Electroconducting Flows, vol. 1, p. 255-260, Aussois, France, 1997.
5. Bartulis A., Kremenetsky V., Shcherbinin E. *The general method of a self-similar solutions construction for 2D MHD flows.* Magnetohydrodynamics, vol. 34, No.2, p.89-104, 1998.
6. Kremenetsky V., Shilova E., Shcherbinin E. *On Magnetohydrodynamic Flows in Convergent and Divergent Diffuser*. Magnetohydrodynamics, Vol. 39, No. 2, p. 179-186, 2003.
7. Kremenetsky V., *On the Self-similar Heat Boundary Layer Problems In Magnetohydrodynamics,* Magnetohydrodynamics, Vol. 37, No. 4, p. 373-378, 2001.
8. Kremenetsky V., *About the Self-similar Heat Boundary Layer Problems In Magnetohydrodynamics,* MATHEMATICAL MODELLING AND ANALYSIS, Vol.7, 2002.
9. Kremenetsky V. *MHD analogues of the Couette flow with suction/injection.* Proc. the 5th International PAMIR Conference "Fundamental and Applied MHD", Vol. I, Ramatuelle, France, September 16-20, 2002.
10. Kremenetsky V., Shcherbinin E. *On some magnetohydrodynamic flows in polar coordinates.* Proc. the 5th International PAMIR Conference "Fundamental and Applied MHD", Vol. II, Ramatuelle, France, September 16-20, 2002.
11. KremenetskyV. *Twisted MHD Jet Flows.* Proc. the 15th Riga and 6th PAMIR Conference on Fundamental and Applied MHD, Vol. I, Jurmala, Latvia, June 27 – July 1, 2005.

The results presented in the thesis were reported at the conferences:

1. 25. RTU zinātniskā un tehniskā konference 1994. g. 19. – 22. aprīlī.
2. Mahyd'95, Jurmala, Latvia,1995.
3. The 3rd International Conference on Transfer Phenomena in Magnetohydrodynamics and Electroconducting Flows, Aussois, France, 1997.
4. 6th International Conference on Mathematical Modelling and Analisys, May 31 – June 2, 2001, Vilnius, Lithuania.
5. 5th International PAMIR Conference "Fundamental and Applied MHD", Vol. I, Ramatuelle, France, September 16–20, 2002.
6. 15th Riga and 6th PAMIR Conference on Fundamental and Applied MHD, Vol. I, Jurmala, Latvia, June 27 – July 1, 2005.

Summary

Section I. Exact solutions in hydrodynamics

In this section, possibilities to transform the equations of hydrodynamics, which are equations with partial derivatives, into ordinary differential equations by introducing a self-similar variable and by separating the variables are considered. Four coordinate systems are used: Cartesian, polar, cylindrical and spherical (the latter two coordinate systems were used in an axially symmetric case). The method of investigation is the following: types of variables and stream functions, which allow to transform the hydrodynamic equations with partial derivatives into ordinary differential equations, are found; then possible formulations of the problem and boundary conditions, which satisfy the derived equations, are considered.

I.i. Short survey of literature

Here a short survey of literature is presented concerned with the solution of hydrodynamic problems in exact formulation in the considered coordinate systems.

I .1. Exact solutions in the Cartesian coordinates

In the Cartesian coordinates it is possible to derive a precise self-similar solution only when choose the self-similar variable and stream function as

$$\eta = y/Bx,\ \psi = A\,f(\eta),$$

and possible flows are described by the equation

$$24\eta f' + 4\eta f'^2 + 12\left(1+3\eta^2\right)f'' + 2\left(1+\eta^2\right)f' f'' + 12\left(1+\eta^2\right)\eta f''' + \left(1+\eta^2\right)^2 f^{IV} = 0 .$$

Should the method of variable decomposition is applied, the stream function should be

$$\psi = g(x)(a\,y + b),$$

and the equation of motion reads

$$a\left(g'g'' + gg'''\right) + \nu\, g^{IV} = 0 .$$

Some solutions of this equation

$$g(x) = -6\nu/ax;\ g(x) = C_1 x + C_2 .$$

The first of the above two solutions has never been met in literature.

I.2. Exact solutions in the polar coordinates

It is shown in the polar coordinate system that only the method of variable separation is applicable. It is possible to derive equations dependent as on the variable φ as on the valuable r, with the stream function in both cases in the form $\psi = R(r)f(\varphi)$.

In case when the equations depend only on the variable φ, the equation of motion turns out

$$f^{IV} + 4f'' + 2f'f'' = 0 .$$

In such formulation the stream function and the velocity component are the following

$$\psi = \nu\, f(\varphi),\ V_r = \nu\, f'/r,\ V_\varphi \equiv 0,$$

that corresponds to radial spreading over or draining of the liquid from a liner source/drain.

In case the equations depend only of the variable r, then $f = a_1 \nu \varphi + C$. At $C = 0$ the equation of motion reads

$$a_1\left[\left(R''' + \frac{R''}{r} - \frac{R'}{r^2}\right)\frac{R}{r} - \left(\frac{R'}{r} + R''\right)\frac{R'}{r}\right] = \frac{R'}{r^3} - \frac{R''}{r^2} + 2\frac{R'''}{r} + R^{IV} .$$

In such formulation

$$\psi = a_1 \nu \varphi R(r),\ V_r = a_1\, \nu\, R/r,\ V_\varphi = -a_1\, \nu\, \varphi R' .$$

Since there is no limitation for the function $f(\varphi)$, so it can be presented as $f'(\varphi) = 0$, and the equation of motion transforms to

$$\frac{R'}{r^3} - \frac{R''}{r^2} + 2\frac{R'''}{r} + R^{IV} = 0,$$

whose solution is an expression like

$$R(r) = Ar^2(\ln r - 1) + Br^2 + C\ln r + D .$$

In this case, $\psi = \mathrm{const}\,R(r)$, where $R(r)$ is determined by the equation solution, and the velocity components are

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\varphi} \equiv 0,\ V_\varphi = -\frac{\partial\psi}{\partial r} = -\mathrm{const}\left[Ar(2\ln r - 1) + 2Br + \frac{C}{r}\right].$$

In such formulation only viscous inertia-free (Stokes) rotational motions of the liquid are possible.

Along with the above cases, other versions of the stream function can exist, which transform the hydrodynamic equations to ordinary differential equations. In some cases, the stream function appearance is determined by the flow geometry, and the method of separation of variables is not always applicable.

I.3. Exact axisymmetric solution in the cylindrical coordinates

Exact solutions have been derived in the cylindrical coordinate system by applying both the self-similar approach and the method of variable separation. The author shows that the transformation of the Navier-Stokes equations is possible only if choose the variables and the stream function in one of the following versions:

1) $\eta = B\,r/z$, $\psi = A\,r\,f(\eta)$, $V_\varphi = C\,\Omega(\eta)/r$;

2) $\eta = B\,z/r$, $\psi = A\,z\,f(\eta)$, $V_\varphi = C\,\Omega(\eta)/z$;

3) $\eta = Br$, $\psi = Azf(\eta)$, $V_\varphi = C\,\Omega(\eta)$;

4) $\psi = a_1 z\,R(r)$, $V_\varphi = \sqrt{a_1 a_6}\,z\,\Omega(r)$;

5) $\psi = a_3 r^2 \Phi(z)$, $V_\varphi = \sqrt{a_3 a_8}\,r\theta(z)$.

Version 3) was derived by the self-similar approach, but in form it is a variable separation. Its difference from version 4) is in different azimuthal velocities.

For each version the author has derived sets of equations, which can be applied without any additional transformations as soon as the type of constants in variables and stream functions is determined.

I.4. Exact axisymmetrical solution in the spherical coordinates

It is shown that in the spherical coordinate system the method of separation of variables is only applicable. It is possible to derive equations dependent **only** on the θ-variable, with the stream function presented in the form $\psi = \nu\,R\,\Phi(\theta)$ and the azimuthal velocity as $V_\varphi = \dfrac{\nu}{R}\Omega(\theta)$. A corresponding set of non-linear ordinary differential equations is

$$\frac{(2+\cos 2\theta)}{\sin^3\theta}\Phi\Phi' + \frac{\operatorname{ctg}\theta}{\sin\theta}\left(1-\frac{4}{\sin^2\theta}\right)\Phi'^2 - \frac{3\operatorname{ctg}\theta}{\sin\theta}\Phi\Phi'' + \frac{3}{\sin\theta}\Phi'\Phi'' +$$

$$+\frac{1}{\sin\theta}\Phi\Phi''' + \frac{3(\cos 3\theta - 3\cos\theta)}{2\sin^3\theta}\Phi' + 4\operatorname{ctg}\theta\;\Phi' + \frac{5}{8}\left(\frac{8}{5}+\frac{5}{\sin^2\theta}\right)\Phi'' - 2\operatorname{ctg}\theta\,\Phi''' + \Phi^{IV} +$$

$$+2\cos\theta\;\Omega^2 + 2\sin\theta\,\Omega\Omega' = 0$$

$$\sin^2\theta\,\Omega'' + \sin\theta\cos\theta\,\Omega' - \sin^2\theta\,\Omega - \Omega + \cos\theta\,\Phi\Omega + \sin\theta\,\Phi F'\Omega' = 0\,.$$

I.5. Conclusions

Here the results of Section I are presented in short.

Section II. Solutions in the boundary layer approximation in hydrodynamics

In this section, the author discusses some possibilities to derive self-similar solutions for the boundary layer problems in hydrodynamics. The study was carried out for the following cases: 1) plane flows in the Cartesian and polar coordinates and 2) axisymmetric flows in the cylindrical and spherical coordinates. As a result, types of variables and stream functions were obtained allowing to transform the Navier-Stokes equations to ordinary equations. In some cases, solutions of these equations were derived and corresponding problems formulated, which are described by these solutions. It has been shown that the boundary layer equations coincide with the exact equations in the polar and spherical coordinates.

II.i. Short survey of literature

In this section a short survey of literature is presented concerned with the solution of boundary layer problems in hydrodynamics in the considered coordinate systems.

II.ii. Preliminary remarks

This section shortly presents the basic statements of the boundary layer theory in hydrodynamics.

II.1. Boundary layer in the Cartesian coordinates

To transform the Navier-Stokes equations into self-similar boundary layer equations it is necessary the self-similar variable and the stream function to present as $\psi = Ax^{1-\beta} f(\eta)$, $\eta = y/\left(Bx^{\beta}\right)$, with the inequality $\beta < 1$ being satisfied. A general equation of motion for plane boundary layer flows has been derived:

$$f^{IV} + (1-\beta)\frac{AB}{\nu} ff''' - (1-3\beta)\frac{AB}{\nu} f'f'' = 0 .$$

To make the equation dimension-free, the coefficients A and B must have the following dimensionalities:

$$[AB] = [\nu] = L^2/T,\ [B] = L^{1-\beta},\ [A] = L^{\beta+1}/T .$$

The method of separation of variables in the boundary layer theory is inapplicable because it is impossible to estimate the orders of terms in the Laplace operator and determine, under which conditions and which terms can be ignored. At $\beta = 0$ the first term in the Laplace operator is automatically turned to zero. This does not contradict to the condition $\beta < 1$, but in fact we have an exact solution, which at the same time is a solution in the boundary layer approximation, and the type of the variable and stream function corresponds to the method of variable separation.

II.2. Hydrodynamic boundary layer in the polar coordinates

The polar system of coordinate does not allow to apply the Prandtl approach because there is no possibility to estimate the orders of terms in the Laplace operator. Therefore, exact solutions must be derived.

II.3. Hydrodynamic axisymmetric boundary layer in the cylindrical coordinates

Within the framework of self-similar approach, the following versions to derive boundary layer solutions are proposed:

1. $\eta = B z^{\beta} r^{\gamma}$, $\psi = A r^{\alpha} f(\eta)$, $V_{\varphi} = C r^{d} \Omega(\eta)$, $E^2 = \dfrac{\partial^2}{\partial z^2}$, $\alpha = \dfrac{\gamma}{\beta} + 2$, $d = 2\dfrac{\gamma}{\beta} + 1$, $\dfrac{\gamma}{\beta} > -1$, $\alpha > 1$, $d > -1$;

2. $\eta = B z^{\beta} r^{\gamma}$, $\psi = A z^{\alpha} f(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $E^2 = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}$, $\alpha = 1$, $d = 1 + \frac{2\beta}{\gamma}$, $\frac{\beta}{\gamma} > -1$, $d > -1$;

3. $\eta = B r^{\gamma}$, $\psi = A z^{\alpha} f(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $E^2 = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}$, $\alpha = 0$, $C = 0$ ($V_{\varphi} = 0$);

4. $\eta = B r^{\gamma}$, $\psi = A z^{\alpha} f(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $E^2 = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}$, $\alpha = 0$, $d = 0$;

5. $\eta = B r^{\gamma}$, $\psi = A z^{\alpha} f(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $E^2 = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}$, $\alpha = 1$, $C = 0$ ($V_{\varphi} = 0$);

6. $\eta = B r^{\gamma}$, $\psi = A z^{\alpha} f(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $E^2 = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}$, $\alpha = 1$, $d = 1$;

7. $\eta = B r^{\gamma}$, $\psi = A z^{\alpha} f(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $E^2 = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}$, $\alpha = 1$, $d = 0$;

8. $\eta = B z^{\beta}$, $\psi = A r^{\alpha} f(\eta)$, $V_{\varphi} = C r^{d} \Omega(\eta)$, $E^2 = \frac{\partial^2}{\partial z^2}$, $\alpha = 2$, $C = 0$ ($V_{\varphi} = 0$);

9. $\eta = B z^{\beta}$, $\psi = A r^{\alpha} f(\eta)$, $V_{\varphi} = C r^{d} \Omega(\eta)$, $E^2 = \frac{\partial^2}{\partial z^2}$, $\alpha = 2$, $d = 1$.

It is shown that some of the above versions allowing to derive boundary layer solutions are at the same time exact solutions. These are versions 3 – 9. The thesis presents as the corresponding sets of self-similar equations as the analysis of possible problems for each version. Formally, four types of self-similar boundary layers have been obtained, two of which are well-known and described in literature, and other two have not been ever derived and interpreter physically. Further, these types of boundary layers have not been studied.

The method of separation of variables in the boundary layer approximation is inapplicable here. More precisely, this method does not allow to estimate the terms' orders in the operator E^2 and ignores one of them in favour of the other. In fact, the considered versions 3 – 9 are themselves the method of variable separation.

II.4. Hydrodynamic axisymmetric boundary layer in the spherical coordinates

The spherical coordinate system does not permit to apply the Prandtl approach because there is no possibility to estimate the orders of terms in the Laplace operator. Therefore, exact solutions must be derived.

II.5. Conclusions

This subsection presents in short the results of Section II.

Section III. Exact solutions in magnetohydrodynamics

This section considers possibilities to derive exact self-similar solutions in magnetohydrodynamics, namely, with account for the influence of electromagnetic forces from external and induced electric and magnetic fields on the conducting liquid. Types of electric and magnetic fields are defined, allowing to transform the magnetohydrodynamic equations to ordinary differential equations. Possible types of self-similar hydrodynamic, electric and magnetic stream functions and magnetohydrodynamic equations have been derived. The non-inductive approximation is also considered, which is frequently used in practical calculations.

III.i. Short survey of literature

In this Section a short survey of literature is presented concerned with the solution of problems in exact formulation in magnetohydrodynamics in the coordinate systems considered here.

III.1. Magnetohydrodynamic equations

This subsection presents general magnetohydrodynamic equations and the boundary conditions for electric and magnetic fields.

III.2. Plane MHD flows in the Cartesian coordinates

It is shown that within the framework of precise approach (with no approximations) the attempt to derive self-similar solutions results in a fact that the only applicable magnetic field does not affect the velocity field. The same can be referred to the electric current passing in the liquid.

If apply the method of variable separation, the hydrodynamic and magnetic stream functions (the electric current will not affect the plane flow) must be chosen as

$$\psi = g(x)(a\,y + b),\ \psi_2 = c\,g_2(x)(a\,y + b).$$

Possible flows and magnetic fields are determined by the equations

$$a\left(g'g'' - gg'''\right) - \frac{ac^2}{\mu_0\rho}\left(g_2'g_2'' - g_2g_2'''\right) + \nu\,g^{IV} = 0\,,$$

$$a\mu_0\sigma\left(g'g_2 - gg_2\right)' + g_2'' = 0\,.$$

These equations have not been met in literature previously.

III.3. MHD flows in the polar coordinates

It is shown that in this coordinate system it is possible to derive equations determined by the variable *r* only, with the stream functions ψ and ψ_2 presented as

$$\psi = a\,\nu\,\varphi\,R(r),\ \psi_2 = a\,k\,\varphi\,R_2(r).$$

III.4. Axisymmetric MHD flows in the cylindrical coordinates

The author shows that to transform the exact Navier-Stokes-Maxwell equations into ordinary differential equations by the self-similar approach and by the method of variable separation, the stream function and the variables must be taken in one of the below forms:

1) $\eta = B\,r/z,\ \psi = A\,r\,f(\eta),\ V_\varphi = C\,\Omega(\eta)/r,\ \psi_1 = N\,f_1(\eta),\ \psi_2 = D\,r\,f_2(\eta)$;

2) $\eta = B\,z/r,\ \psi = A\,z\,f(\eta),\ V_\varphi = C\,\Omega(\eta)/z,\ \psi_1 = N\,f_1(\eta),\ \psi_2 = D\,z\,f_2(\eta)$;

3) $\eta = Br,\ \psi = Azf(\eta),\ V_\varphi = C\,\Omega(\eta),\ \psi_1 = N\,f_1(\eta),\ \psi_2 = D\,z\,f_2(\eta)$;

4) $\psi = a_1 z\,R(r),\ V_\varphi = \sqrt{a_1 a_6}\,z\,\Omega(r),\ \psi_1 = g\,z\,R_1(r),\ \psi_2 = d\,z\,R_2(r)$;

5) $\psi = a_3 r^2\Phi(z),\ V_\varphi = \sqrt{a_3 a_8}\,r\theta(z),\ \psi_1 = g\,r^2\,\Phi_1(z),\ \psi_2 = d\,r^2\Phi_2(z)$.

These results are in a good agreement with those in [43], but the performed analysis differs from [43], moreover, version 2) is not mentioned in [43]. The thesis presents the ready-to-use sets of ordinary equations for each version.

III.5. Axisymmetric MHD flows in the spherical coordinates

To derive exact solution in the spherical coordinate system, the stream functions must be presented as

$$\psi = \nu R \Phi(\theta), \;\; \psi_1 = \nu \sqrt{\rho/\mu_0}\, \Phi_1(\theta), \; \psi_2 = \nu \sqrt{\rho \mu_0}\, R \Phi_2(\theta), \; V_\varphi = \frac{\nu}{R} \Omega(\theta).$$

In the thesis the ordinary equations related to the θ-variable have been transformed by the method of separation of variables.

III.6. Non-inductive approximation

Often at considering the magnetohydrodynamic flows the so-called non-inductive approximation is used, which considers electric currents induced by the motion of conducting liquid subject to a magnetic field, but does not consider the induced magnetic fields.

The general principle of the non-inductive approximation is the following [43], [71], [72]): the magnetic stream function ψ_2 is presented as a series $\psi_2 = \sum_{n=0}^{\infty} \beta^n \psi_{2n}$ with a small non-dimensional parameter - the Batchelor number $\beta = \nu \sigma \mu_0$, which under the terrestrial conditions is of order $\beta \approx 10^{-6} \div 10^{-7}$, where ψ_{20} is the undisturbed by the liquid motion external magnetic field. Only the series terms ψ_{20} and ψ_{21} affect the liquid motion, the others have a higher power in β and can be ignored because of the smallness of β.

III.6.1. Non-inductive approximation in the Cartesian coordinates for plane flows

In the non-inductive approximation and using the self-similar approach, the magnetic field, similar to that in the exact formulation, does not affect the velocity field.

It is shown that the application of the method of variable separation in the non-inductive approximation there is only one version, which permits to formulate the magnetohydrodynamic problem, i.e., the flow in a magnetic field with a neutral point:

$$\psi_2 = (a_1 x + a_2) c\, y .$$

The equation of motion in this case reads

$$\frac{c^2}{\rho \mu_0}\left(a_1^2 g - a_1 (a_1 x + a_2) g' + (a_1 x + a_2)^2 g''\right) + a\left(g'g'' - g g'''\right) + \nu g^{IV} = 0 .$$

III.6.2. Non-inductive approximation in the polar coordinates

The non-inductive approximation provides some possibilities to obtain applicable magnetic fields (similar to in the Cartesian coordinates the electric current does not affect the velocity field):

1) $\psi_{20} = (A\varphi + B)(C \ln r + D)$,

2) $\psi_{20} = \left(A \sin n\varphi + B \cos n\varphi\right)\left(C r^{n} + D r^{-n}\right)$

3) $\psi_{20} = \left(A e^{n\varphi} + B e^{-n\varphi}\right)\left(C \sin(n \ln r) + D \cos(n \ln r)\right)$

Such magnetic fields require no special passing of the electric current j_z because the above solutions have been derived from the equation $0 = \nabla^2 \psi_{20} = (\mathrm{rot} B)_z = \mu_0 (j)_z$. Examples of some possible fields are considered here and illustrated in Fig.III.1.

While deriving equations dependent on the φ-variable, it turns out that the ***only possible*** problem about a really radial diverging (diffuser) and converging (confuser) MHD flow in the azimuthal magnetic field induced by a linear current circuit can be considered, which is described by the equation

$$f^{IV} + 4f'' + 2f'f'' - \mathrm{Ha}^2 f'' = 0 .$$

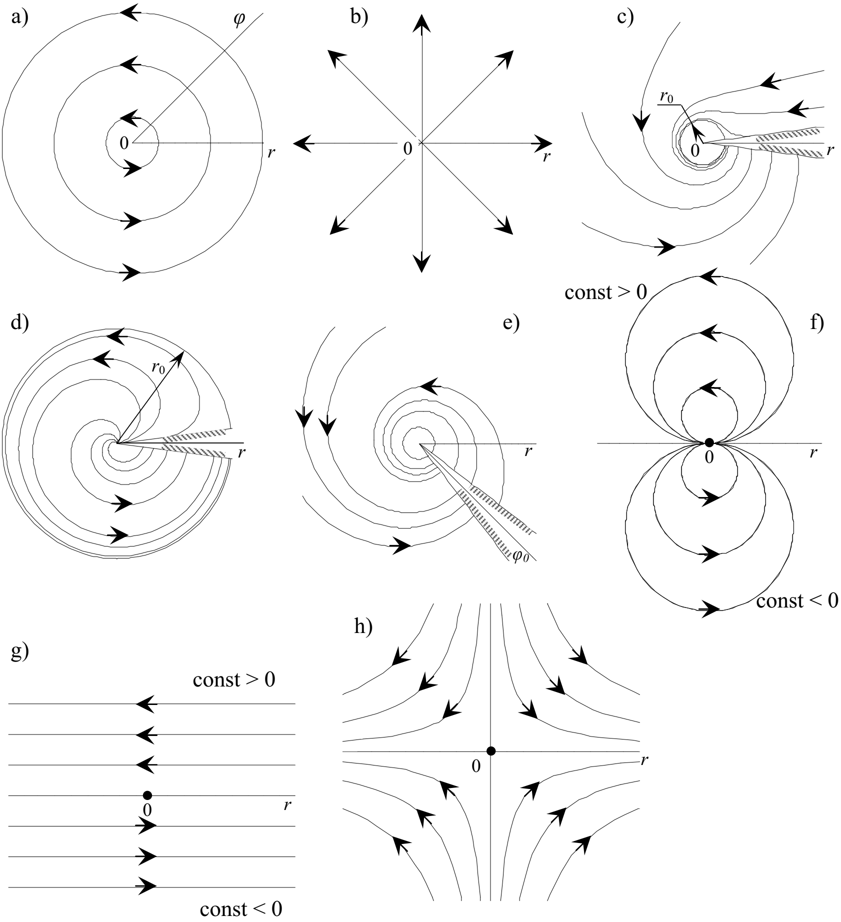

Fig.III.1. Configurations of magnetic force lines for different separation constants n: a), b), c), d), e) – n = 0; f), g) – n = 1; h) – n = 2.

While deriving equations depending only of r, we have that the stream function should be taken as $\psi = a\,\nu\,\varphi\,R(r)$, $\psi_{20} = m\ \varphi\,(C\ \ln r + D)$, and the equation of motion reads as

$$E^2(E^2R) - a\left[\frac{R}{r}(E^2R)' - \frac{R'}{r}E^2R\right] =$$

$$= -\frac{\sigma m^2}{\rho\nu}\left\{\frac{C\ln r + D}{r}\left[C\left(\frac{R}{r^2}\right)' - \left\{\frac{1}{r}R'(C\ln r + D)\right\}' + \frac{C}{r^2}R'\right] - \frac{C^2R}{r^4}\right\}.$$

The equation becomes simpler if assume $C = 0$:

$$E^2(E^2R) - a\left[\frac{R}{r}(E^2R)' - \frac{R'}{r}E^2R\right] = \mathrm{Ha}^2\,\frac{1}{r}\left(\frac{R'}{r}\right)',$$

where $\dfrac{\sigma m^2 D^2}{\rho\nu} = \mathrm{Ha}^2$. Moreover, the external magnetic field is pure radial:

$$B_r = mD/r, \quad B_\varphi = 0$$

and in this formulation it does not forcely affect the flow field because of the presence of the azimuthal velocity V_φ.

The method of separation of variables is not the only way to derive Navier-Stokes-Maxwell ordinary differential equations. The hydrodynamic stream functions can be presented as

$$\psi = B\,f\,(\varphi) + l\,(\varphi)\ln r\,.$$

It this case, to derive an ordinary differential equation relative to φ, we must accept $l\,(\varphi)$ = const. Then,

$$V_r = B\,f'(\varphi)/r, \quad V_\varphi = -l\,(\varphi)/r = -\mathrm{const}/r\,.$$

III.6.3. Non-inductive approximation in the cylindrical coordinates in an axisymmetric case

The same classes of solutions as in the full formulation are considered in the non-inductive approximation and applicable magnetic fields are defined:

1) $\eta = Br/z$, $\psi = Ar\,f(\eta)$, $V_\varphi = C\Omega(\eta)/r$, $\psi_1 = N\,f_1(\eta)$, $\psi_{20} = Dr\,f_2(\eta)$.

The magnetic field is defined by the function

$$f_2(\eta) = C_1\sqrt{1+\left(\frac{B}{\eta}\right)^2}\,,$$

which corresponds to the field of a magnet, whose one pole is a symmetry axis *Oz*, the other pole is a cone-like surface.

2) $\eta = Bz/r$, $\psi = Az\,f(\eta)$, $V_\varphi = C\Omega(\eta)/z$, $\psi_1 = N\,f_1(\eta)$, $\psi_{20} = Dz\,f_2(\eta)$.

The magnetic field is defined by the function

$$f_2(\eta) = C_1 - C_2\,\frac{\sqrt{B^2+\eta^2}}{B^2\,\eta}\,,$$

which at $C_1 = 0$ by accuracy to the sign coincides with the magnetic stream function from version 1.

3) $\eta = Br$, $\psi = Az\,f(\eta)$, $V_\varphi = C\Omega(\eta)$, $\psi_1 = N\,f_1(\eta)$, $\psi_{20} = Dz\,f_2(\eta)$.

The magnetic field is defined by the function

$$f_2(\eta) = C_1\,\eta^2/2 + C_2 = C_1\,B^2r^2/2 + C_2\,.$$

At $C_1 \neq 0$, $C_2 = 0$ the induction compounds of such a field correspond to a magnetic field with a neutral point. At $C_1 = 0$, $C_2 \neq 0$ the magnetic field lines are straight lines perpendicular to the z-axis that corresponds to a radial magnetic field, whose intensity decreases farther from the symmetry axis.

4) $\psi = a_1 z\,R(r)$, $V_\varphi = \sqrt{a_1 a_6}\,z\,\Omega(r)$, $\psi_1 = g\,z\,R_1(r)$, $\psi_{20} = d\,z\,R_2(r)$.

The magnetic field is defined by the function

$$\psi_{20} = d\,z\left(C_1\,r^2/2 + C_2\right).$$

This magnetic function coincides with the magnetic function of version 3.

5) $\psi = a_3 r^2 \Phi(z)$, $V_\varphi = \sqrt{a_3 a_8} r\theta(z)$, $\psi_1 = g\, r^2\, \Phi_1(z)$, $\psi_{20} = d\, r^2 \Phi_2(z)$.
The magnetic field is defined by the function

$$\psi_{20} = d\, r^2 (C_1 z + C_2).$$

This is a magnetic field with a neutral point.

No other versions exist.

III.6.4. Non-inductive approximation in the spherical coordinates in an axisymmetric case

In the non-inductive approximation the stream function remain the same as in exact formulation, and the magnetic stream function is defined by the expression

$$\psi_2 = DR(C_2 - C_1 \cos\theta).$$

At $C_2 = 0, C_1 \neq 0$, the magnetic field described by this formula is a radial field, and at $C_1 = 0, C_2 \neq 0$, it corresponds to a field, whose one pole is a symmetry axis *Oz*, the other pole is a cone-like surface.

III.7. Conclusions

In this subsection one can find the results of Section III presented in short.

Section IV. Boundary layers in magnetohydrodynamics

This section considers possibilities to derive boundary layer magnetohydrodynamic solutions by choosing an electromagnetic term in exact formulation and in the non-inductive approximation. The study is performed with reference to the results obtained in Sections II and III. In accordance with these results, no polar and spherical coordinate systems are considered (exact equations must be considered in these coordinate systems) as well as the method of variable separation.

IV.i. Short survey of literature.

This subsection presents a short survey of papers concerned with the theory of boundary layer in magnetohydrodynamics in the considered in the thesis coordinate systems.

IV.1. Plane MHD layers in the Cartesian coordinates

To modify the equations, the self-similar variable and the stream functions must be presented as

$$\eta = y/\delta(x),\ \ \delta(x) = Bx^\beta,\ \ \psi = Ax^\alpha f(\eta),\ \ \psi_2 = Dx^d f_2(\eta),$$

where the power induces are related as

$$\alpha = d = 1 - \beta,\ \beta < 1.$$

There are two ways to make the equation of induction self-uniform: 1) $\beta = 1/3$, $K \neq 0$; and 2) $K = 0$. The first way means that a direct current passes in the liquid along the O*z*-axis, but this is possible only at one predetermined value of β. In the second case, no current passes.

If $\alpha = 1,\ \beta = 0$, we have here the separation of variables. Moreover, this solution is an exact solution because no approximation is used. Yet, at the same time, it is a boundary layer solution.

IV.2. Axisymmetric MHD boundary layers in the cylindrical coordinates

It is shown here that in the cylindrical coordinate system there are seven possible ways to transform the MHD boundary layer equation in exact formulation into ordinary differential equations (corresponding set of equations are presented in the thesis, they are numbered similarly).

1. $\eta = B z^{\beta} r^{\gamma}$, $\psi = A r^{\alpha} f(\eta)$, $\psi_2 = D r^{\alpha} f_2(\eta)$, $V_{\varphi} = C r^{d} \Omega(\eta)$, $\psi_1 = N r^{d+1} f_1(\eta)$, $E^2 = \dfrac{\partial^2}{\partial z^2}$, $\alpha = \dfrac{\gamma}{\beta} + 2$, $d = \dfrac{2\gamma}{\beta} + 1$, $\dfrac{\gamma}{\beta} > -1$, $\alpha > 1$, $d > -1$;

2. $\eta = B z^{\beta} r^{\gamma}$, $\psi = A z^{\alpha} f(\eta)$, $\psi_2 = D z^{\alpha} f_2(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $\psi_1 = N z^{p} f_1(\eta)$, $E^2 = r \dfrac{\partial}{\partial r} \dfrac{1}{r} \dfrac{\partial}{\partial r}$, $\alpha = 1$, $d = 1 + \dfrac{2\beta}{\gamma}$, $p = 1 + \dfrac{\beta}{\gamma}$, $\dfrac{\beta}{\gamma} > -1$, $d > -1$;

5. $\eta = B r$, $\psi = A z^{\alpha} f(\eta)$, $\psi_2 = D z^{\alpha} f_2(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $\psi_1 = N z^{p} f_1(\eta)$, $E^2 = r \dfrac{\partial}{\partial r} \dfrac{1}{r} \dfrac{\partial}{\partial r}$, $\alpha = 1$, $C = 0$ ($V_{\varphi} = 0$), $p = 1$;

6. $\eta = B r$, $\psi = A z^{\alpha} f(\eta)$, $\psi_2 = D z^{\alpha} f_2(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $\psi_1 = N z^{p} f_1(\eta)$, $E^2 = r \dfrac{\partial}{\partial r} \dfrac{1}{r} \dfrac{\partial}{\partial r}$, $\alpha = 1$, $d = 1$, $p = 1$;

7. $\eta = B r$, $\psi = A z^{\alpha} f(\eta)$, $\psi_2 = D z^{\alpha} f_2(\eta)$, $V_{\varphi} = C z^{d} \Omega(\eta)$, $\psi_1 = N z^{p} f_1(\eta)$, $E^2 = r \dfrac{\partial}{\partial r} \dfrac{1}{r} \dfrac{\partial}{\partial r}$, $\alpha = 1$, $d = 0$, $p = 0$;

8. $\eta = B z$, $\psi = A r^{\alpha} f(\eta)$, $\psi_2 = D r^{\alpha} f_2(\eta)$, $V_{\varphi} = C r^{d} \Omega(\eta)$, $\psi_1 = N r^{p} f_1(\eta)$, $E^2 = \dfrac{\partial^2}{\partial z^2}$, $\alpha = 2$, $C = 0$ ($V_{\varphi} = 0$), $p = 2$;

9. $\eta = B z$, $\psi = A r^{\alpha} f(\eta)$, $\psi_2 = D r^{\alpha} f_2(\eta)$, $V_{\varphi} = C r^{d} \Omega(\eta)$, $\psi_1 = N r^{p} f_1(\eta)$, $E^2 = \dfrac{\partial^2}{\partial z^2}$, $\alpha = 2$, $d = 1$, $p = 2$.

IV.3. MHD boundary layers in the non-inductive approximation

To derive self-similar MHD boundary layer solution in the non-inductive approximation, we refer to and use the results obtained previously in this Section and the theory discussed in subsection III.6.

IV.3.1. Plane MHD boundary layers in the Cartesian coordinates in the non-inductive approximation

In the non-inductive approximation, it is necessary to consider additionally the electric stream function ψ_1:

$$\eta = y/\delta(x),\ \delta(x) = Bx^{\beta},\ \psi = Ax^{\alpha} f(\eta),\ \psi_1 = N x^{\gamma} f_1(\eta),\ \psi_{20} = Dx^{d} f_2(\eta),$$

with the relation between the powers like

$$\alpha = 1 - \beta,\ d = \alpha = 1 - \beta,\ \gamma - \text{any},\ \beta < 1.$$

Here a unique set of self-similar solutions is presented. Analysis of the induction equation shows that to make it self-similar there are two ways: 1) $\beta = 1/3,\ K \neq 0$, and 2) $K = 0$. In the first case, we get a problem, where the magnetic field is determined by

$$\psi_{20} = Dx^{1-\beta}\left(\frac{KB^2}{2D}\eta^2 + C_1\eta + C_2\right) = \frac{K\,y^2}{2} + \frac{DC_1\,y}{B}x^{1/3} + DC_2 x^{2/3}.$$

This stream function describes the combination of three magnetic fields: the first term corresponds to a magnetic field, whose force lines are parallel to the x-axis and the induction of which increases linearly with the increase of y; the second term corresponds to a magnetic field, whose force lines are described by an equation $y = const\,x^{-1/3}$ (hyperbolic curves) and thicken at the y-axis and the induction decreases further from the coordinate origin (this magnetic field is similar to a magnetic field with a neutral point); the third term corresponds to a magnetic field, whose force lines are parallel to the y-axis and the induction decreases further from *Oy*.

In the second version of the induction equation at K = 0, the magnetic stream function is determined by the expression

$$f_2(\eta) = C_1\eta + C_2,\; \psi_{20} = Dx^{1-\beta}\left(\frac{C_1\,y}{B\,x^{\beta}} + C_2\right) = \frac{DC_1}{B}y\,x^{1-2\beta} + C_2 x^{1-\beta},$$

describing a magnetic field, which is a combination of the following fields: the first term corresponds to a field, the force lines of which are described by the equation $y = const\ x^{2\beta-1}$; due to the values of $\beta < 1$, it is possible to produce different fields: at $\beta = 0$ a magnetic field with a neutral point, at $\beta = 1/2$ a uniform longitudinal magnetic field $B_x = const,\, B_y = 0$; the second term corresponds to a field, the force lines of which are oriented parallel to the y-axis ($x = const$) $B_x = 0,\, B_y = C_2(1-\beta)x^{\beta}$.

It is possible to derive a solution, which, by accepting $\alpha = 1,\ \beta = 0$, simultaneously is an exact solution and a boundary layer solution. In this case, the self-similar solution coincides in appearance with a solution derived by the method of variable separation. The only applicable in this case field is the field with a neutral point.

IV.3.2.Axisymmetric MHD boundary layers in the cylindrical coordinates in the non-inductive approximation

There are nine versions to transform equations in the non-inductive approximation. Both the sets of equations and applicable magnetic and electric fields (in some versions) have been derived.

1. The magnetic stream function

$$\psi_{20} = D\,r^{\alpha}\left(\tilde{C}_1\beta\eta^{\frac{1}{\beta}} + \tilde{C}_2\right) = D\,r^{2+\frac{\gamma}{\beta}}\left(\tilde{C}_1\beta B^{\frac{1}{\beta}}\,z\,r^{\frac{\gamma}{\beta}} + \tilde{C}_2\right).$$

At $\tilde{C}_1 = 0$, the force lines are determined by a condition $r^{2+\gamma/\beta} = const$. This field is parallel to the symmetry z-axis. Herewith, only at $\gamma/\beta = 0$ the magnetic field is uniform; at other $\gamma/\beta > -1$ values the density of magnetic force lines depends on γ/β. At $\tilde{C}_2 = 0$, the magnetic force lines are located at the surfaces of rotation, whose generators are determined by a power function $z = const\,/\,r^{2+2\gamma/\beta}$ ($2 + 2\gamma/\beta > 0$), moreover, as γ/β increases, the force lines thicken at the symmetry axis, and the r-component of the field does not depend on z. For example, at $\gamma/\beta = -1/2$ the magnetic force lines are located on surfaces, the generators of which are hyperboloids $z = const/r$. At $\tilde{C}_1 \neq 0$ and $\tilde{C}_2 \neq 0$, we get different combinations of these two versions.

2. The magnetic stream function is the following:

$$\psi_{20} = D\,z\left(\frac{\gamma\,B^{\frac{2}{\gamma}}\widetilde{C}_1}{2}z^{\frac{2\beta}{\gamma}}r^2 + \widetilde{C}_2\right).$$

At $\widetilde{C}_1 = 0$, the force lines of magnetic field $\psi_{20} = 0$ correspond to a definitely radial magnetic field. At $\widetilde{C}_2 = 0$, due to the sign in the expression $1 - 2\beta/\gamma$, the force lines are either on paraboloids ($0 > 1 - 2\beta/\gamma > -1$) or on hyperboloids ($1 - 2\beta/\gamma > 0$) of rotation; at $1 - 2\beta/\gamma = 0$, we have a uniform axial magnetic field $B_z = const$.

3. The magnetic and electric stream functions are defined by the expressions

$$f_1(\eta) = \frac{\widetilde{C}_1\eta^2}{2} + \widetilde{C}_2,\; f_2(\eta) = \frac{\hat{C}_1\eta^2}{2} + \hat{C}_2,$$

moreover, to consider the MHD problem, we must assume $\widetilde{C}_1 = 0$. Hence, no electric current passes in the liquid, and the magnetic stream function reads

$$\psi_{20} = D\,z\left(\frac{\hat{C}_1 B^2}{2}r^2 + \hat{C}_2\right).$$

The magnetic field described by this formula is a combination of a radial field and a field with a neutral point.

4. The magnetic stream function is the same as in version 3.

5. The magnetic stream function is the same as in version 3. the electric stream function is defined as

$$f_1(\eta) = \frac{\widetilde{C}_1\eta^2}{2} + \widetilde{C}_2.$$

Formally, one can derive an equation to define the hydrodynamic stream function *f*:

$$2\widetilde{C}_2 f + \eta\left(\widetilde{C}_1\,\eta^2/2 + \widetilde{C}_2\right)f' = 0,$$

a solution of which a function ($\overline{C}$ is the constant of integration)

$$f(\eta) = \left(\widetilde{C}_1 + \frac{2\,\widetilde{C}_2}{\eta^2}\right)\overline{C}.$$

Yet, generally, it is impossible to learn whether this function, being substituted into the equation, is the solution because the values of the constants *A, B, D* and the constants of integration must be known.

6. The magnetic stream function is the same as in version 3.

7. The magnetic stream function is the same as in version 3. In this case, the azimuthal rotation and the electric current do not affect the axial and radial velocities, but have to “tune up” with them.

8. The magnetic stream function is defined by the expression

$$\psi_{20} = D\,C_1 B\,z\,r^2 + D\,C_2\,r^2,$$

that corresponds to the magnetic field with a neutral point. The relation between the magnetic and electric stream functions is found:

$$f_1(\eta) = \overline{C}\,f_2(\eta),$$

hence, the electric stream function is of the same type as the magnetic one:

$$f_1(\eta) = \hat{C}_1\eta + \hat{C}_2 = \hat{C}_1 B\,z + \hat{C}_2.$$

Substituting this expression into the equation of motion formally yields the hydrodynamic stream function *f*:

$$f(\eta) = 0,$$

that is insignificant. A non-trivial solution is yielded if set $\hat{C}_1 = 0$. Then, there are two ways:

either $\hat{C}_2 = 0$ or $f_2 = const$. The first version means that no electric current passes through the liquid ($\psi_1 \equiv 0$), the second one means that the liquid flow is subject to a uniform axial magnetic field $\psi_{20} = DC_2 r^2$, $B_z = 2DC_2$, $B_r = 0$.

9. The magnetic stream function is the same as in version 8. In contrast to version 8, in the framework of this version, problems with the azimuthal (rotational) velocity can be considered.

Versions 5 – 9 are not only solutions in the boundary layer approximation, but also exact solutions and correspond to the method of separation of variables.

IV. Conclusions

The results obtained in Section IV are presented in short.

Section V. Some self-similar problems for hydrodynamics and magnetohydrodynamics

In this Section, some hydrodynamic and magnetohydrodynamic problems are considered, which can be solved in the self-similar formulation in the approach proposed in Sections I-IV of the thesis. As examples, MHD analogues of well-known hydrodynamic problems are chosen as well as modifications of the problems with introduced suction and/or injection through solid walls.

V.1. MHD flows in diffuser and confuser

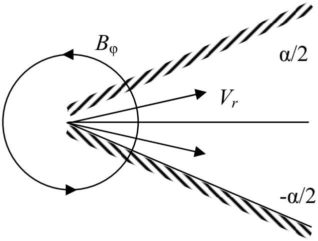

Fig. V.1. Schematic presentation of the flow in a flat diffuser in an azimuthal magnetic field.

The Hamel flow with a velocity field $V_r = \nu f'/r$, $V_\varphi = 0$ in a magnetic field induced by a linear current circuit $B_\varphi = -C/r$ is considered. Let us consider an approximate solution of this problem in the polar coordinates in the non-inductive approximation, for large Hartmann numbers. The separation of variables is inapplicable in this case, and the stream functions have to be chosen with reference to flow peculiarities.

Schematically, the flow is illustrated in Fig.V.1. The hydrodynamic and magnetic stream functions with account for the type of velocity components and magnetic field are the following:

$$V_r = \nu f'/r\,,\ V_\varphi = 0\,,\ \psi = \nu f(\varphi),$$

$$B_r = 0\,,\ B_\varphi = -l/r\,,\ \psi_{20} = l \ln r\,.$$

The equation of motion is

$$f^{IV} + 4f'' + 2f'f'' - \mathrm{Ha}^2 f'' = 0\,,$$

where $\mathrm{Ha}^2 = \sigma l^2 / \rho\nu$, the no-slip boundary conditions on the diffuser (confuser) walls with the angle of opening α:

$$f'(\pm\alpha/2) = 0\,,$$

the symmetry of velocity profile

$$f''(0) = 0$$

and the integral condition of flowrate conservation

$$\int_{-\alpha/2}^{\alpha/2} V_r r\, d\varphi = \nu \int_{-\alpha/2}^{\alpha/2} f' d\varphi = \pm Q\,.$$

If denote the magnitude Q/ν through the Reynolds number *Re*, the condition of conservation reads:

$$\int_{-\alpha/2}^{\alpha/2} f'\,d\varphi = \pm\ \mathrm{Re},$$

where (+) is referred to a diffuser flow and (–) to a confuser one.

An approximate solution at large Ha^2 can be derived by a known method of exchange of the sought function and variable, according to $f(\varphi)=\mathrm{Ha}^{\mathrm{m}}F(\eta), \quad \varphi=\mathrm{Ha}^{\mathrm{n}}\eta$. In order to retain identity of the expression for the radial velocity, we set f' and F' as $f'(\varphi)=\mathrm{Ha}^{\mathrm{m}-\mathrm{n}}F'(\eta), \quad \mathrm{m}=\mathrm{n}$ and, assuming $\mathrm{n}=-1$ to keep balance between the viscous and electromagnetic terms of the equation of motion, we have

$$F^{IV}-F''+(4F''+2F'F'')/\mathrm{Ha}^2=0.$$

Searching for the approximate solution (165) as a series in reverse powers of Ha^2 $F=\sum_{\mathrm{n}=0}^{\infty}(\mathrm{Ha}^{-2})^{\mathrm{n}}F_{\mathrm{n}}$ yields a set of equations for sequential approximations of equation solutions.

At transition in the derived solutions to the limiting case $\mathrm{Ha}\to\infty$, the velocity on the symmetry axis $\varphi=0$ with an accuracy of term order of Ha^{-3} is

$$f'(0)/(\pm\mathrm{Re})=\frac{1}{\alpha}\left(1+\frac{2}{\alpha\,\mathrm{Ha}}+\frac{4}{\alpha^2\,\mathrm{Ha}^2}+\frac{8}{\alpha^3\,\mathrm{Ha}^3}+\frac{5}{3}\frac{(\pm\mathrm{Re})}{\alpha^2\,\mathrm{Ha}^3}+\frac{4}{\alpha\ \ \mathrm{Ha}^3}\right).$$

If Re is of order $\alpha\,\mathrm{Ha}^2$, the term related to Re in this expression is a correction in case of large Re and Ha; if $\alpha\,\mathrm{Ha}^2$ >> Re, then, the normalized in Reynolds number radial velocity in the flow core is $1/\alpha$ for both diffuser and confuser flows, and only in a boundary layer of Ha^{-1} thickness it decreases to zero. This agrees with the result in [67] and also follows from the calculation of velocity profiles

$$f'=f_0'+f_1'/\mathrm{Ha}^2,$$

the expression for which is presented in two series terms with solution for F_0' and F_1'.

There are only few differences in the increasing of the normalized velocity on the symmetry axis towards its limiting value $1/\alpha$ at $\mathrm{Ha}\to\infty$ in the diffuser and confuser flows. In the diffuser flow (+ Re) the axial velocity increases towards its limiting value monotonically with the increase of Ha. If compare to the confuser flow (–Re), the axial velocity achieves its maximum value with the increase of Ha at predetermined α and Re and at a Ha defined by solving a square equation $\mathrm{Ha}^2+4\mathrm{Ha}/\alpha+(12-5\mathrm{Re}/2)/\alpha^2+6/\alpha=0$ (e.g., if $\alpha=\pi/3$, Re = 100, we have $\left[f'(0)/(-\mathrm{Re})\right]\big|_{\max}$ = 1.0545 at Ha = 13.1), and further, with the increase of Ha value it approaches its limiting values (see Fig. V.2). Though at a predetermined Re the maximum deviation of the velocity from limiting on the axis is comparatively insignificant, this could reason the non-monotony of the velocity profile found from the analysis of confuser flow in general, which is very pronounced at increased Re and predetermined Ha values.

Friction is found from

$$f''=\mathrm{Ha}\,F''=\mathrm{Ha}\,F_0''+F_1''/\mathrm{Ha},$$

and at large Ha and normalization in $\pm\mathrm{Re}$ it reads:

$$f''(-\alpha/2)/(\pm\mathrm{Re})=\frac{\mathrm{Ha}}{\alpha}\left(1+\frac{2}{\alpha\,\mathrm{Ha}}\right)-\frac{(\pm\mathrm{Re})}{\alpha^2\mathrm{Ha}}\left(\frac{2}{3}+\frac{1}{\alpha\,\mathrm{Ha}}\right).$$

As seen from the limiting formula, the reduction of the angle of opening α results in friction increase at a predetermined Re and $\mathrm{Ha} >> |\mathrm{Re}|/\alpha^2$ in the both diffuser (+Re) and confuser (– Re) flows. From the same expression it follows that at a predetermined Ha in the diffuser flow the increase of Re causes a friction decrease, therefore, at Re > $\mathrm{Re}_{\mathbf{cr}}$ the flow can break off the wall. In the confuser flow, the increase of Re is always accompanied by an increased friction on the wall.

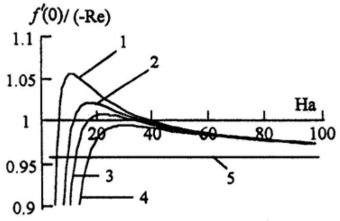

Fig.V.2. Variation of the normalized in Re velocity on the symmetry axis determined by Re and Ha values in the confuser flow. Curve 1 – Re = – 100, 2 – Re = – 200, 3 – Re = – 300, 4 – Re = – 400, 5 – asymptotic velocity values at large Ha.

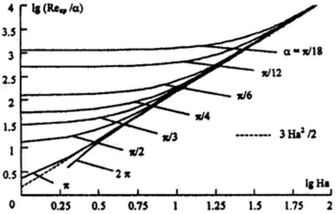

Fig.V.3. Critical Re value vs. Ha at different values of the angle of diffuser opening in a logarithmic scale. Dashed lines illustrate the dependence at Ha >> 1.

Of interest is to define the critical Reynolds number determined by α and Ha, at which the diffuser flow is just about to break off (as known for the confuser flow, there is no flow break off from solid walls). The moment of break off is determined by zero friction on the wall:

$$\mathrm{Re}_{\mathbf{cr}} = 3\alpha\mathrm{Ha}^2(2+\alpha\mathrm{Ha})/(3+2\alpha\mathrm{Ha}) \approx 3\alpha\mathrm{Ha}^2/2\,.$$

This result crucially differs from the one obtained in [67] ($\mathrm{Re}_{\mathbf{cr}}$ = 6Ha) in a way that $\mathrm{Re}_{\mathbf{cr}}$ quadratically, not linearly, depends on the Hartmann number and, besides, $\mathrm{Re}_{\mathbf{cr}}$ is determined also by the angle of diffuser opening α. A more detailed dependence of $\mathrm{Re}_{\mathbf{cr}}/\alpha$ on Ha is illustrated in Fig.V.3. As seen, at large Ha, the curves at all values of α become asymptotic $\mathrm{Re}_{\mathbf{cr}}/\alpha \approx 3\mathrm{Ha}^2/2$. At moderate Ha values, these data are hardly trustable because the problem formulation assumes Ha >> 1.

V.2. MHD flow with suction or injection on a plate in an azimuthal magnetic field in the non-inductive approximation

Such flow illustrated in Fig.V.4 is characterized by setting the hydrodynamic stream function as

$$\psi = Bf(\varphi) + C\ln r\,,$$

by a velocity field

$$V_r = \pm\mathrm{Re}f'(\varphi)/r\,,\; V_\varphi = -(\pm\lambda)\,\nu/r\,,$$

a magnetic field

$$B_r = 0\,,\; B_\varphi == -l/r$$

and by the magnetic stream function

$$\psi_{20} = l\ln r\,,$$

And the following equation serves as an equation of motion

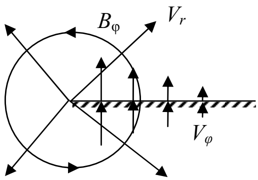

Fig.V.4. Schematic presentation of flow on a plate with suction/injection in an azimuthal magnetic field.

$$f^{IV}+4f''\pm 2\ \mathrm{Re}\, f'f''\pm\lambda f'''-\mathrm{Ha}^2 f''=0,$$

where $\mathrm{Ha}^2=\sigma l^2/\rho\nu$.

The considered problem is of the same type as the previous one, when the stream functions are chosen due to the flow peculiarities. In this case, the separation of variables is inapplicable.

The Reynolds number Re is defined from the expressions described in the end of subsection III.6.2, and the signs determine the flow type: (+) for a diffuser (diverging) flow (V_r>0) and (–) for a confuser (converging) flow ($V_r<0$).

The problem is considered under the following conditions: 1) there is a flat wall (plate), on which some suction (+λ) or injection (–λ) intensity of the liquid is set; 2) it is supposed that at some predetermined Re value, i.e., the source (Re > 0) or the draining (Re < 0) intensity of the liquid, placed at the coordinate origin $r=0$, Ha >> 1. And the situation itself can be presented as follows: there is a vortical thread at the coordinate origin, which, due to [70], initiates azimuthal rotation $V_\varphi=-(\pm\lambda)\,\nu/r$, so on the impenetrable solid surface $\varphi=0,\ \ 0\le r=\infty$ suction of the liquid takes place (+ λ) if assume $\varphi\ge 0$, i.e., above the surface, and injection of the liquid through the same surface if assume $\varphi\le 0$, i.e., below this surface.

For the function f', the set of boundary conditions is the following:

$$f'(0)=0,\quad f'(\pi/2)=1,\quad f''(\pi/2)=0,$$

if consider the radial velocity f' alone.

If make substitutions $f(\varphi)=F(\eta)/\mathrm{Ha}$, $\varphi=\eta/\mathrm{Ha}$ in the equation of motion and termwise divide by Ha^3, it becomes

$$F^{IV}+(4F''\pm 2\mathrm{Re}F'F'')/\mathrm{Ha}^2\pm\lambda F'''/\mathrm{Ha}-F''=0.$$

The second of the above-mentioned conditions, in this case, supposes a possibility to derive an approximate solution as a series

$$F(\eta)=\sum_{\mathrm{n}=0}(1/\mathrm{Ha})^{\mathrm{n}}\,F_{\mathrm{n}}(\eta).$$

Substituting the series into the equation and equalizing coefficients in equal powers $1/\mathrm{Ha}$ yields a set of equations for sequential approximate solutions of the equation.

Figs.V.5, V.6 illustrate the results of velocity profile calculation for the diffuser and confuser flows at different values of the suction (injection) parameter λ and different Hartmann numbers from the formula

$$f'(\varphi)=F'(\eta)=F_0'(\eta)+F_1'(\eta)/\mathrm{Ha}+F_2'(\eta)/\mathrm{Ha}^2$$

using the obtained solutions.

In general, the growth of Ha provides flattening of the velocity profile in both flows (Fig.V.5b,d; Fig.V.7b,d).

For the diffuser flow in at a predetermined Ha (Fig.V.5a), the suction of the liquid from the plate surface prevents the flow break off from the surface, but its increase testifies to a formation of a reverse flow in a distant from the surface zone (curves 4, 5). At some predetermined λ, the increase of Ha causes smoothening of the velocity profile (Fig.V.5b). The same phenomenon takes place in the confuser flow in Fig.V.6a,b. The liquid injection assists to the flow breaking off from the surface in the diffuser flow (Fig.V.5c).

As to the confuser flow, the breaking off from the surface does not take place even at comparatively small Ha and large injection values, but as $|\,\lambda\,|$ increases, there is a tendency to and even appears a zone with a reverse flow in an area distant from the surface (Fig.V.6c). At some predetermined λ, the increase of Ha causes smoothening of the velocity profile in both flows (Fig.V.5d and V.6d).

The specific idea about the aforesaid with respect to flow breakaway at the solid surface can be obtained, if we turn to the large Hartman numbers (Ha >> 1). A simple expression for friction on the wall reads, then

$$f''(0) = \mathrm{Ha}\left(F_0''(0) + \frac{1}{\mathrm{Ha}}F_1''(0) + \frac{1}{\mathrm{Ha}^2}F_2''(0)\right) = \mathrm{Ha}\left[1 + \frac{1}{\mathrm{Ha}}\frac{(\pm\lambda)}{2} + \frac{1}{\mathrm{Ha}^2}\left(\frac{\lambda^2}{8} - \frac{2}{3}(\pm\mathrm{Re}) - 2\right)\right],$$

from which it follows that at liquid suction from the plate surface ($+\lambda$) the friction on the wall increases, preventing in such a way the boundary layer breakaway in the diffuser flow (+Re). The increase of Re in the same flow decreases the friction on the wall, therefore, if the following condition is satisfied

$$\mathrm{Re} > \mathrm{Re}_{\mathbf{cr}} = 3\mathrm{Ha}^2/2 + 3\mathrm{Ha}\,(\pm\lambda)/4 + 3\lambda^2/16 - 3,$$

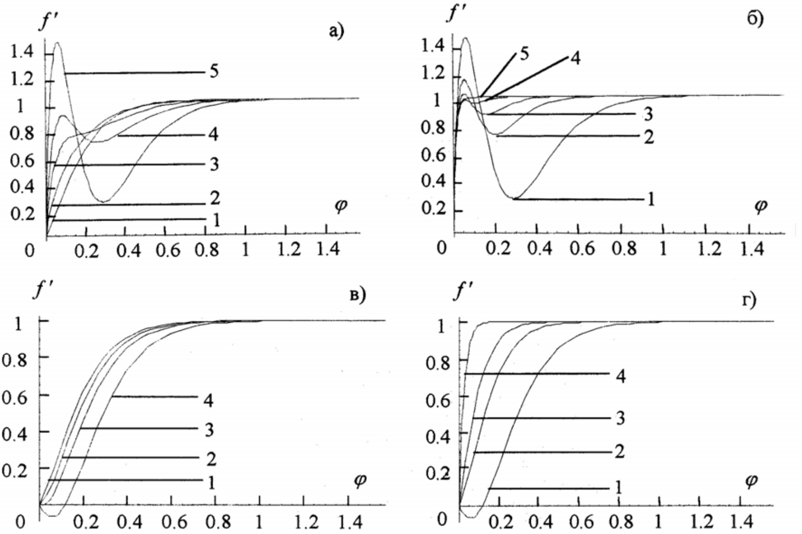

Fig.V.5. Distribution of the normalized in Re radial velocity in the azimuthal magnetic field in the diffuser flow at suction from the plate surface (a, b) or at injection through the same surface (в, г) for predetermined Re = 100. The curves in the figures correspond to a) Ha = 10, $\lambda = 1$ (curve 1), $\lambda = 10$ (2), $\lambda = 20$ (3), $\lambda = 30$ (4), $\lambda = 50$ (5); b) $\lambda = 50$, Ha = 10 (curve 1), Ha = 15 (2), Ha = 20 (3), Ha = 30 (4), Ha = 50 (5); c) Ha = 10, $\lambda = -1$ (curve 1), $\lambda = -5$ (2), $\lambda = -10$ (3), $\lambda = -20$ (4); d) $\lambda = -20$, Ha = 10 (curve 1), Ha = 15 (2), Ha = 20 (3), Ha = 50 (4).

the flow breaks off. With no suction or injection ($\lambda = 0$), at Ha >> 1 this formula agrees with the formula for $\mathrm{Re}_{\mathbf{cr}}$ derived in the previous subsection, differing only by the absence of the multiplier α at Ha^2 that is determined by the difference in boundary conditions.

The critical Re number $\mathrm{Re}_{\mathbf{cr}}$ is a function of two parameters Ha and λ. In the diffuser flow at suction of the liquid ($+\lambda$) $\mathrm{Re}_{\mathbf{cr}}$ monotonically increases as the both parameters increase. If the liquid is injected ($-\lambda$) and the parameter $|\lambda|$ is predetermined, there is a local extremum (minimum) determined by $\mathrm{Re}_{\mathbf{cr}}$ (Ha) at point Ha = $|\lambda|/4$ (herewith, $\mathrm{Re}_{\mathbf{cr}} = 3\lambda^2/16$). If Ha is predetermined, the same extremum determined by $\mathrm{Re}_{\mathbf{cr}}(\lambda)$ appears at point $|\lambda| = 2\mathrm{Ha}$ (herewith, $\mathrm{Re}_{\mathbf{cr}} = 3\mathrm{Ha}^2/4$).

This conclusion is illustrated in Fig.V.7. In the area above the corresponding curves $f''(0) < 0$, i.e., we have a breaking away flow, if, for a chosen value of ($-\lambda$), Re > $\mathrm{Re}_{\mathrm{cr}}$. Respectively, the zone of parameters Re and ($-\lambda$), which is below these curves, denotes a no-breaking off flow ($f''(0) > 0$). At suction of the liquid ($+\lambda$), the value of $\mathrm{Re}_{\mathbf{cr}}$ monotonically increases with the increase of both parameters Ha and Re. In the confuser flow (–Re), neither increase of suction ($-\lambda$) is capable to initiate a pre-breaking off situation. It should be noted that in such flow at predetermined Ha and Re the dependence $f''(0)$ on the strength of suction is minimum at $\lambda = 4\mathrm{Ha}$. Herewith, $f''(0) = \mathrm{Ha}^2 + 2\mathrm{Re}/3$.

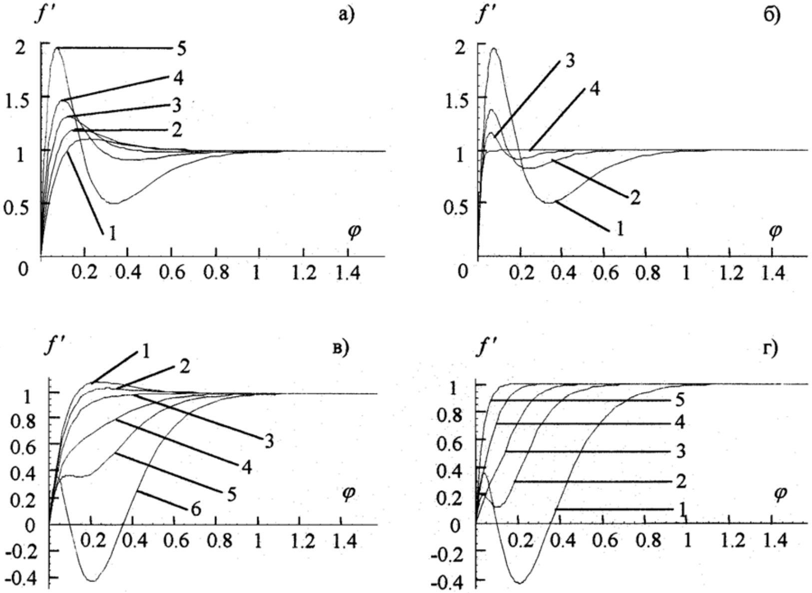

Fig.V.6. Distribution of the normalized in Re radial velocity in the azimuthal magnetic field in the confuser flow at suction from the plate surface (a, b) or at injection through the same surface (c, d) for predetermined Re = -100. The curves in the figures correspond to a) Ha = 10, $\lambda = 1$ (curve 1), $\lambda = 10$ (2), $\lambda = 20$ (3), $\lambda = 30$ (4), $\lambda = 50$ (5); b) $\lambda = 50$, Ha = 10 (curve 1), Ha = 15 (2), Ha = 20 (3), Ha = 50 (4); c) Ha = 10, $\lambda = -1$ (curve 1), $\lambda = -5$ (2), $\lambda = -10$ (3), $\lambda = -20$ (4), $\lambda = -30$ (5); $\lambda = -50$ (6); d) $\lambda = -50$, Ha = 10 (curve 1), Ha = 15 (2), Ha = 20 (3), Ha = 30 (4), Ha = 50 (5).

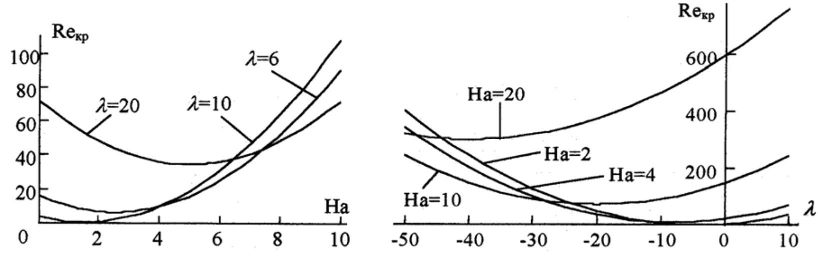

Fig.V.7. Some features of the dependence $\mathrm{Re}_{\mathbf{cr}}$ (λ, Ha) in the diffuser flow. Values of λ in the left-hand part correspond to $\left|-\lambda\right|$, i.e., to liquid injection.

V.3. MHD flow with suction or injection on a plate in a radial magnetic field

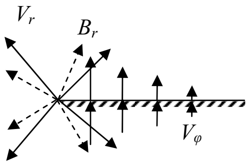

Fig.V.8. Schematic presentation of a flow on a plate with suction/injection in a radial magnetic field.

The situation illustrated in Fig.V.8 shows that the transformation of the velocity field might take place mainly due to an electromagnetic force related to the azimuthal velocity and radial field.

The hydrodynamic stream function, the velocity components and the boundary conditions are the same as in the previous task; the magnetic stream function and the magnetic field are described by the expressions

$$\psi_{20} = m\varphi(C\ln r + D),\ B_r = l/r, B_\varphi = 0.$$

The equation of motion

$$f^{IV} + 4f'' \pm 2\mathrm{Re}\, f' f'' \pm \lambda f''' + 2\mathrm{Ha}^2(\pm\lambda)/(\pm\mathrm{Re}) = 0,$$

where $\mathrm{Ha}^2 = \sigma l^2/\rho\nu$, after transformations of $f(\varphi) = F(\eta)/\mathrm{Ha}$, $\varphi = \eta/\mathrm{Ha}$ becomes

$$F^{IV} + [4F'' + 2(\pm\mathrm{Re})F'F'']/\mathrm{Ha}^2 + (\pm\lambda)F'''/\mathrm{Ha} + 2\alpha/\mathrm{Ha} = 0,$$

with $\alpha = (\pm\lambda)/(\pm\mathrm{Re})$. It must be solved with the following boundary conditions

$$F'(0) = 0, \quad F'(\mathrm{Ha}\,\pi/2) = 1, \quad F''(\mathrm{Ha}\,\pi/2) = 0.$$

The problem is solved by the same method as in subsection V.2.

The velocity profiles are found from the same formula as in the previous subsection:

$$f'(\varphi) = F'(\eta) = F_0'(\eta) + F_1'(\eta)/\mathrm{Ha} + F_2'(\eta)/\mathrm{Ha}^2.$$

Let us define conditions, at which the flow break-off from the overflown surface is possible. For the friction on the wall $\varphi = 0$ $(\eta = 0)$, we have:

$$f''(0) = \mathrm{Ha}\left(F_0''(0) + F_1''(0)/\mathrm{Ha} + F_2''(0)/\mathrm{Ha}^2\right) =$$
$$= \frac{4}{\pi} - \frac{\pi}{3} + \frac{(\pm\lambda)}{3} + \frac{\pi\lambda^2}{36} - \frac{\lambda^2\pi^3\mathrm{Ha}^2}{114(\pm\mathrm{Re})} - \frac{(\pm\lambda)\,\pi^2\,\mathrm{Ha}^2}{12\,(\pm\mathrm{Re})} - \frac{\pi\,(\pm\mathrm{Re})}{10}.$$

To define the critical $\mathrm{Re}_{\mathbf{cr}}$ a square equation is derived:

$$\mathrm{Re}_{\mathbf{cr}}^{\mathbf{2}} - \mathrm{bRe}_{\mathbf{cr}} + \mathrm{c} = 0,$$

where $b = (\lambda/3 + 4/\pi - \pi/3 + \pi\lambda^2/36)10/\pi$, $c = (\mathrm{Ha}\,\pi/2)^2(\lambda\cdot 10/3\pi + 5\lambda^2/18)$. Presenting in this way the expressions for *b* and *c*, we assume that λ and $\mathrm{Re}_{\mathbf{cr}}$ can have both positive and negative values.

Solving the square equation $\mathrm{Re}_{\mathbf{cr}} = \mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{c}}/2$ yields that if $c < 0$, the sign (+) in this solution corresponds to the diffuser flow ($\mathrm{Re}_{\mathbf{cr}} > 0$), and (–) to the confuser one. If $c > 0$ and $b^2 - 4c \geq 0$, then $\mathrm{Re}_{\mathbf{cr}}$ is always positive, and this case can correspond to the diffuser flow alone at $b > 0$. If $b < 0$, we deal with the confuser flow.

Let us analyze first the confuser flow, when $\mathrm{Re}_{\mathbf{cr}}$ can exist only at $\lambda^2 < -12\lambda/\pi$. At $\lambda > 0$ (suction) this conditions is not satisfied, therefore, no flow break off might takes place at suction of the liquid from the flown over plate surface. If $-12/\pi < \lambda < 0$, it occurs that at injection of the liquid there is a zone of $(-\lambda)$ and (-Re) values, where the breaking off confuser flow is possible. Calculations of the $\mathrm{Re}_{\mathbf{cr}}(-\lambda)$ dependence in the confuser flow ($\mathrm{Re}_{\mathbf{cr}} < 0$) for some values of Ha^2 are shown in Fig.V.9a. This dependence at a predetermined Ha turns out non-monotonic, and when λ approaches $\lambda = 0$ and $\lambda = -12/\pi$, $\mathrm{Re}_{\mathbf{cr}} \to 0$. The zone of no-breaking off flow is below the corresponding curves 1-4 and λ-axis away from the $-12/\pi < \lambda < 0$ range. In the zone above these curves towards the λ-axis the flow breaks off.

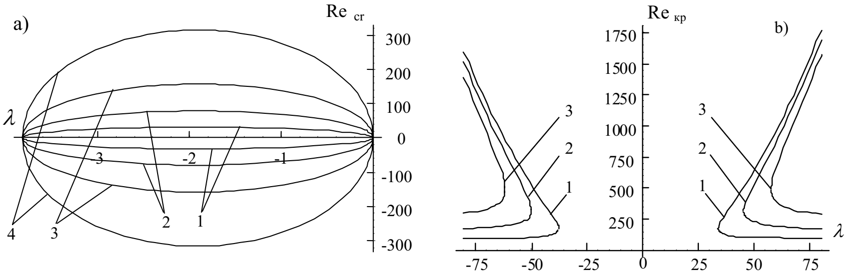

Fig.V.9. Re_{cr} vs. λ: a) within $0 > \lambda > -12/\pi$, and b) in a range of large $|\lambda|$. Top of Fig.V.9a denotes diffuser flow; bottom - confuser flow. Curves are numbered according to Ha: 20 (curve 1), 50 (2), 100 (3), 200 (4). b) diffuser flow, Ha = 6 (curve 1), 8 (2), 10 (3).

For the diffuser flow with the same λ variation range, the formation of no-breaking off flow zone is possible in a radial magnetic field in spite of injection (with no magnetic field, any injection most likely breaks the flow away). The no-breaking off flow is illustrated at the top of Fig.V.9a, below the corresponding curves 1-4. Here $f''(0) > 0$. In the zone above these curves, the diffuser flow breaks off ($f''(0) < 0$). When approach the boundaries of this λ variation range, $\mathrm{Re}_{\mathbf{cr}} \to 0.719$.

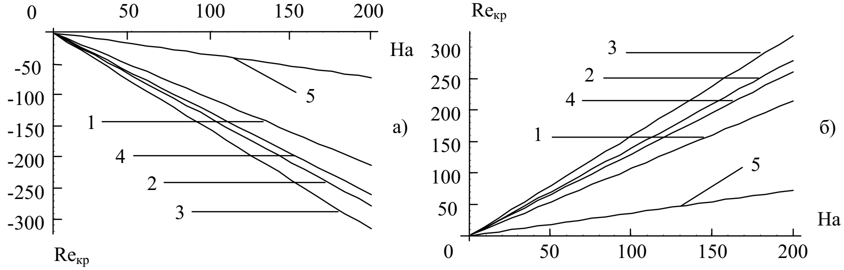

Fig.V.10. Re_{cr} vs. Ha within $0 > \lambda > -12/\pi$ for the confuser a) and diffuser б) flows. λ values: -0.6 (straight line 1), -1 (2), $-6/\pi$ (3), -3 (4), $-12/\pi + 0.05$ (5).

Dependencies Re_{cr} on Ha in the same λ variation range are shown in Fig.V.10 and are almost linear for both confuser (Fig.V.10a) and diffuser (Fig.V.10b) flows. The zone with no break off flow lies below the corresponding straight lines. The obtained results are thoroughly analyzed and interpreted.

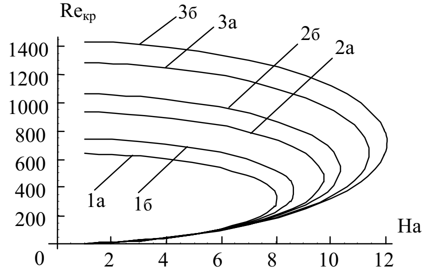

Fig.V.11. Re_{cr} vs. Ha at injection (a) or suction (b) of the liquid and at $|\lambda| = 50$ (curve 1), 60 (2), 70 (3).

The most interesting thing is that the solution at rather large $|\lambda|$ again predicts the formation of no-breaking off flow zone (left-hand part of Fig.V.9b) to the left from the corresponding curves. Formally, this can be explained in a way that at $|\lambda| > 0.979$ the coefficient b is positive, and at $b^2 - 4c > 0$ the Re_{cr} value becomes positive. It seems to be related to the fact that an additional flow initiated by the rotor of electromagnetic force, which is in proportion as to $D^2 A^2 \approx \mathrm{Ha}^2$ as

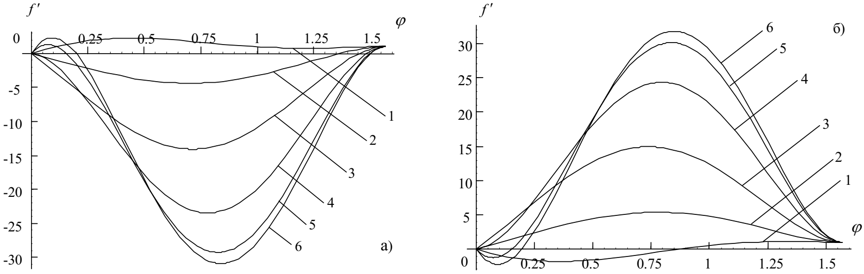

Fig.V.12. Profiles of radial velocity in a) confuser and b) diffuser flows at |Re| = 125 and Ha = 100. Curve enumeration corresponds to $\lambda = -0.5$ (curve 1), $\lambda = -1$ (2), $\lambda = -6/\pi$ (3), $\lambda = -3$ (4), $\lambda = -12/\pi+0.05$ (5), $\lambda = -4$ (6).

to the strength of injection $|\lambda|$, prevents the flow break-off. On the other hand, the injection benefits to the flow break-off from the solid surface. A complicated behaviour of these counteracting factors determines the final state of the flow. If predetermine Ha, then, as follows from Fig.V.11, $\mathrm{Re}_{\mathrm{cr}}$ increases with the growth of $|\lambda|$, i.e., the additional flow dominates over the injection. The same takes place at predetermined $|\lambda|$ and increasing Ha.

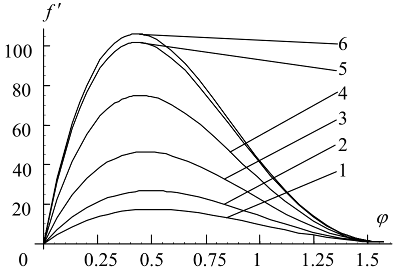

Fig.V.13. Distribution of the radial velocity at Ha = 100, Re = –125 within the range of positive λ: $\lambda = 0.5$ (curve 1); 1 (2); $6/\pi$ (3); 3 (4); $12/\pi + 0.05$ (5); 4 (6).

No sufficient explanation has been found for the zone of breaking off flow within the range of parameters $|\lambda|$ and Ha below the corresponding curves in Fig.V.9b and Fig.V.11. The only thing that can be emphasized is that in this zone $b^2 - 4c < 0$, and, probably, the series terms should be involved for further analysis.

Let us analyze the radial velocity field. In the confuser flow the velocity field is determined by three parameters (Re, Ha, λ) that complicates its analysis greatly. Therefore, we consider only the influence of the parameter λ at injection or suction of the liquid on the distribution of the radial velocity with predetermined Re and Ha. The most interesting features of this distribution are expected within the injection parameter range $12/\pi < \lambda < 0$, and the values of Ha and Re are the following: Ha = 100, Re = –125.

Calculations of f' are illustrated in Fig.V.12. Additionally, the figure shows profiles f' for the diffuser flow as well, which also show the break-off zones.

A common feature of the flow with injection (both confuser and diffuser) in a radial magnetic field is that the radial velocity considerably exceeds its value in the most of the flow zone at $\varphi = \pi / 2$ (at infinite distance from the flown over surface), if compared to the case of azimuthal magnetic field (see subsection V.2). However, most surprising is the influence of the radial magnetic field on the confuser flow. For the sign (–) (confuser) we then have $V_r = -\nu\,|\,\mathrm{Re}\,|\,f'/r$, and, since in the most of the flow $f' < 0$ (for curves 2-6 in Fig.V.12a), in this area $V_r > 0$, i.e., the liquid moves opposite to the main flow. This can be explained only by the action of $\mathrm{rot}\boldsymbol{F}_{\mathrm{em}}$, which drives an additional flow opposite to the main liquid flow.

In the area with positive λ (suction) the confuser flow is always no breaking-off, yet, the radial velocity values in the near-wall zone greatly exceed its value in infinity

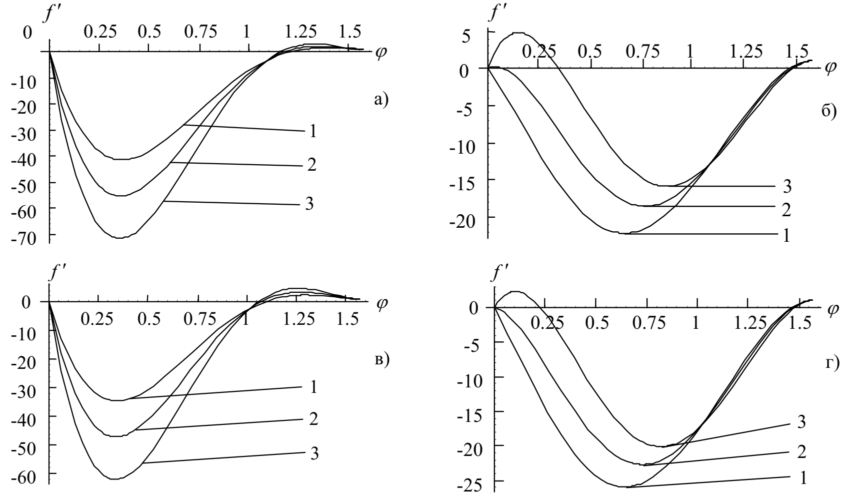

Fig.V.14. Distribution of the radial velocity in the diffuser flow at Ha = 10 in the area of large $|\lambda|$ and at Re = 125 (a, в), Re = 500 (б, г). a) and b) correspond to positive λ; c) and d) to negative λ. Curve enumeration corresponds to $|\lambda| = 50$ (curve 1), 60 (2), 70 (3).

$f'(\pi/2) = 1$ (Fig.V.13).

Some words about the diffuser flow at large $|\lambda|$, to which dependencies Re_{cr} ($\pm\lambda$) correspond (see Fig.V.9b). With some predetermined Ha, the behaviour of the radial velocity field is the following: the near-wall flow zone at relatively small Re values (Fig.V.14a) has a flow running opposite to the main flow, moreover, the stronger suction ($\lambda > 0$), the stronger the reverse flow. At large Re, the increase of suction results in an opposite effect (Fig.V.14b), therefore, the no-breaking off flow might also be driven (curve 3 in Fig.V.14b). Similar features takes place in the radial velocity field at injection ($\lambda < 0$) of the liquid (Fig.V.14c,d).

The Reynolds values 125 and 500 have been chosen because (see Fig.V.9б and Fig.V.11) at Ha = 10, Re = 125 corresponds to the breaking off flows at any $|\lambda|$, and at Re = 500 the flow can be as breaking off as no-breaking off that is determined by the value of $|\lambda|$. The pre-breaking off state of the flow is determined by $\lambda \approx 60$ and $\lambda \approx -65$.

V.4. Circular MHD analogue of the Couette flow in the non-inductive approach

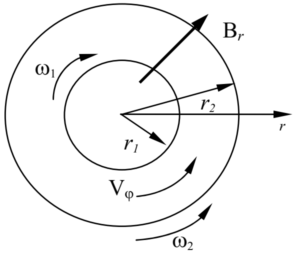

Fig.V.15. Schematic presentation of circular MHD Couette flow.

Let us consider a circular MHD analogue of the Couette flow, which schematically is shown in Fig.V.15. From the hydrodynamic viewpoint, it means that the hydrodynamic stream function reads (15)

$$\psi = kR(\bar{r}),$$

where $\bar{r} = r/r_2$ (r_2 is defined below), $k = \text{const}$ that corresponds to the set velocity field

$$V_r = 0, V_\varphi = -kR'/r_2 .$$

The magnetic stream function reads (99)

$$\psi_{20} = (A\varphi + B)D,$$

i.e., the external magnetic field is expressed as:

$$B_r = A\,D / r_2\,\bar{r},\ \ B_\varphi = 0\,.$$

It follows from the type of the hydrodynamic stream function that only circles $\bar{r} = \text{const}$ can serve as boundaries of the flow area, where ψ is constant and magnetic force lines $\psi_{20} =$ const are rays $\varphi = \text{const}$.

So, the problem formulation implies a flow between two co-axial cylindrical walls of radii r_1 and r_2 $\left(r_1 < r_2\right)$,which in a general case rotate with different angular velocities ω_1 and ω_2 in arbitrary directions, in a radial magnetic field (Fig.V.15).

The equation of motion reads:

$$E^2(E^2 R) = \text{Ha}^2\,\frac{1}{\bar{r}}\left(\frac{R'}{\bar{r}}\right)'\ ,$$

where $\text{Ha}^2 = \sigma A^2 D^2 / \rho\nu = \sigma\, B_0^2 r_2^2 / \rho\nu$, hence, the ratio AD/r_2 is a typical magnetic field B_0 value.

The boundary conditions are $V_\varphi(r_1) = \omega_1 r_1 = \omega_1 r_2 \bar{r}_1 = -kR'(\bar{r}_1)/r_2$ and $V_\varphi(r_2) = = \omega_2 r_2 = -k\,R'(1)/r_2$. If assume $-\,k/r_2 = \omega_1 r_2$, i.e., $k = -\ \omega_1 r_2^2$, the above conditions become $R'(\bar{r}_1) = \bar{r}_1,\ \ R'\,(1) = m,$ where $m = \omega_2 / \omega_1$. Besides, the integral equation of flowrate conservation in circular cross-sections must be used $\varphi = \text{const}$:

$$Q = \int_{r_1}^{r_2} V_\varphi d\,r = -\frac{k}{r_2}\,r_2 \int_{\bar{r}_1}^{1} R' d\,\bar{r} = \omega_1 r_2^2 \int_{\bar{r}_1}^{1} R' d\,\bar{r}\ .$$

If introduce a mean velocity $V_{\varphi\,\text{cp}} = (\omega_1 r_1 + \omega_2 r_2)\ /\ 2\,,$ we have $Q = (\omega_1\, r_1 + \omega_2\, r_2)\ (r_2 - r_1)\ /\ 2$ and the integral equation reads $R\,(1) - R\,(\bar{r}_1) = = (\bar{r}_1 + m)\,(1 - \bar{r}_1)\ /\ 2\,.$

So, the equation of motion must be solved at

$$R'\,(\bar{r}_1) = \bar{r}_1,\quad R'\,(1) = m,\quad R\,(1) - R\,(\bar{r}_1) = (\bar{r}_1 + m)(1 - \bar{r}_1)\ /\ 2\,.$$

In the following, bars above non-dimensional values are omitted.

The equation of motion can be once integrated; the derived equation allows a $R' = F$ order drop. To transform this equation to an Euler equation, its right-hand part must be omitted by using some partial solution. Here of importance is whether Ha = 0 or Ha ≠ 0.

If Ha = 0, we have a partial solution $F_1 = A\,r \ln r\ /\ 2$, and by substituting $F = F_1 + \varPhi$ the equation transforms to an Euler equation, hence, a common solution is

$$R = A(2r^2 \ln r - r^2)/8 + Br^2/2 + C\ln r + D\,.$$

If Ha ≠ 0, a partial solution is $F_1 = -Ar/\text{Ha}^2$ and, by substituting $F_1 = -Ar/\text{Ha}^2 + \varPhi$, we have an Euler equation with a general solution like:

$$R = -\frac{A}{2\,\text{Ha}^2}r^2 + \frac{B}{1+\sqrt{1+\text{Ha}^2}}\,r^{1+\sqrt{1+\text{Ha}^2}} + \frac{C}{1-\sqrt{1+\text{Ha}^2}}\,r^{1-\sqrt{1+\text{Ha}^2}} + D\,.$$

Boundary conditions for integration constants at Ha = 0 provide:

$$B = \frac{1}{1 - r_1^2}(m - r_1^2 + \frac{A}{2} r_1^2 \ln r_1)\,,\ \ C = \frac{r_1^2}{1 - r_1^2}(1 - m - \frac{A}{2}\ \ln r_1)\,,\ \ A = \frac{4\ (1-m)\ r_1}{2r_1 \ln r_1 - (1 - r_1^2)}\,.$$

The same conditions at Ha ≠ 0 also provide an exact solution.

Velocity profiles at Ha ≠ 0 are displayed in Fig.V.16. At small Ha numbers, the distribution of V_φ over the radius is almost linear and, as Ha increases, it becomes S-shaped (Fig.V.16a) that is typical of a plane MHD Couette flow [46] at any ratio of angular velocities $m = \omega_2\ /\ \omega_1$ (Fig.V.16b). Only at m = 1 the radial magnetic field does not affect in any way the quasi-solid-body rotation of the liquid. Velocity profiles in this problem explicitly show the formation of narrow boundary layers at the rotating walls. Herewith, neither conditions are set in the equations, i.e., exact equations are solved. This example shows that the boundary layer solutions can be derived from the solutions of exact equations by tending the problem parameters towards some critical values. In this case, the formation of boundary layers is ensured as by increasing the Hartmann number (magnetic field increase) as the rate of cylinder rotation.

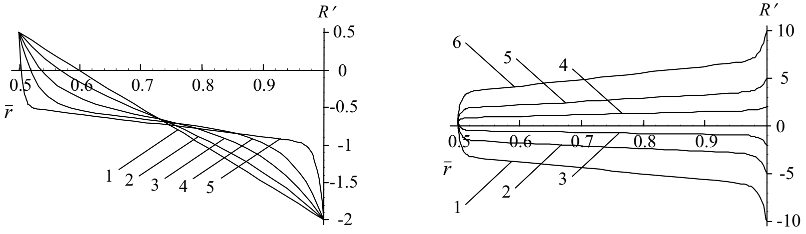

Fig.V.16. Azimuthal velocity distribution in the circular MHD Couette flow at $r_1 = 0.5$: a) $m = -2$ and Ha = 1 (curve 1), 5 (2), 10 (3), 20 (4), 100 (5); b) Ha = 100 and $m = -10$ (curve 1), –5 (2), –2 (3), 2 (4), 5 (5), 10 (6).

For the friction on the walls $\tau_w = \mu \, \partial V_\varphi / \partial r \big|_{r_1, r_2} = -\, \mu \, k \, R'' / r_2^2 = \mu \, \omega_1 R''$, the following exact formulae can be derived:

at Ha = 0

$$R''(r_1) = \frac{r_1^{\,4} - 2r_1^{\,3} + 2r_1^{\,2} + 2r_1 + 2m(r_1^{\,3} - r_1^{\,2} - 2r_1 - 1) - 2r_1(r_1^{\,2} - 1)\ln r_1 + 1}{(r_1^{\,2} - 1)(r_1^{\,2} - 2r_1 \ln r_1 + 1)},$$

$$R''(1) = \frac{2r_1(r_1^{\,3} - r_1^{\,2} + r_1 + 1) - m(r_1^{\,4} - 2r_1^{\,3} + 2r_1^{\,2} + 2r_1 + 1) - 2m r_1(r_1^{\,2} - 1)\ln r_1}{(r_1^{\,2} - 1)(r_1^{\,2} - 2r_1 \ln r_1 + 1)};$$

at Ha ≠0

$$R''(r_1) = \frac{(N-1)\, r_1^{\,N+1} + (N+1)\, r_1^{\,N} - (N+1)\, r_1 - \mathrm{Ha}^2 (m-1)(r_1^{\,N} + 1) - N + 1}{(N-1)\, r_1^{\,N+1} + (N+1)\, r_1^{\,N} - (N+1)\, r_1 - N + 1},$$

$$R''(1) = \frac{\mathrm{Ha}^2 r_1 (r_1^{\,N} + 1) + m\,((1-N) N\, r_1^{\,N+1} + (N+1)\, r_1^{\,N} - (1+N) N\, r_1 - N + 1)}{(N-1)\, r_1^{\,N+1} + (N+1)\, r_1^{\,N} - (N+1)\, r_1 - N + 1}.$$

At large Ha and with account for $r_1 < 1$ the latter two formulae are simplified:

$$R''(r_1) = 1 + (m-1)\,\mathrm{Ha}/(1 + r_1), \qquad R''(1) = m + (m-1)\, r_1\, \mathrm{Ha}/(1 + r_1),$$

the second terms in which determine the magnetic field contribution into the friction on the walls. As seen, in the circular analogue of the Couette flow, the friction on the walls is proportional at the first power of Ha similar to the plane analogue of this flow [46].

V.5. Circular MHD flow between rotating cylinders with suction/injection in the non-inductive approach

The problem considered in subsection V.4 can be generalized for a case when suction/injection of the same liquid, which is between two rotating cylinders, takes place through the cylindrical surfaces. In this situation, along with the external radial magnetic field $B_r = A / r$, as in V.4, an azimuthal magnetic field $B_\varphi = B / r$ can be used, which can be induced by a linear current circuit on the symmetry axis or by an electric current uniformly distributed over the inner cylinder and directed along the z-axis. The above magnetic fields agree with the magnetic stream function

$$\psi_{20} = A\varphi + B \ln r .$$

The velocity field components in an assumption of uniform suction/injection distribution on the cylindrical surfaces and their independence of the coordinate φ must be determined by a more general that in V.4 type of the hydrodynamic stream function

$$\psi = kR(\bar{r}) + C\varphi .$$

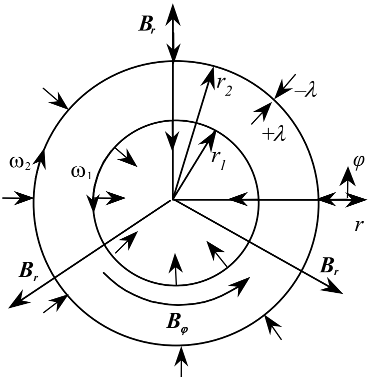

Fig.V.17. Schematic presentation of the MHD flow with suction/injection.

Then

$$V_r = \frac{C}{r_2}\frac{1}{\bar{r}}, \quad V_\varphi = -\frac{k}{r_2} R'(\bar{r}),$$

where $\bar{r} = r / r_2$ (the de-dimensionalization has been done over the radius of the outer cylinder).

Schematically, this flow is illustrated in Fig.V.17.

If additionally assume $C = (\pm\lambda)\nu$, where λ is the parameter of injection (sign (+)) or suction (sign (–)) through the inner cylinder, $k = \nu$, $A = r_2 B_0$, $B = r_2 B_1$, where B_0 and B_1 are the typical values of radial and azimuthal magnetic fields, respectively, then the equation of motions reads (the bar above the non-dimensional magnitudes is omitted):

$$r^4 R^{IV} + 2r^3 R''' - r^2 R'' + rR' - (\pm\lambda)\left(2r^3 R''' + r^2 R'' - rR'\right) -$$
$$-2(\pm\lambda)\left(\pm\overline{\mathrm{Ha}}^2\right) + \mathrm{Ha}^2\left(r^2 R'' - rR'\right) = 0 .$$

Here, $\mathrm{Ha}^2 = \sigma B_0^2 r_2^2 / \rho\nu$ is a Hartmann number derived from the typical value B_0 of the radial magnetic field, and $\overline{\mathrm{Ha}}^2 = \sigma B_0 B_1 r_2^2 / \rho\nu$ is a Hartmann number derived from the typical values of radial (B_0) and azimuthal (B_1) magnetic fields. The choice of sign in front of $\overline{\mathrm{Ha}}^2$ in the equation of motion is determined by mutual orientation of the magnetic field components, while the sign in front of Ha^2 is determined from the fact that the direction of the radial magnetic field itself has no influence because of the quadratic parameter Ha in the equation of motion.

The boundary conditions for the discussed problem are:

$$V_\varphi\Big|_{r=r_1} = r_1\omega_1, \; V_\varphi\Big|_{r=r_2} = r_2\omega_2; \; V_r\Big|_{r=r_1} = (\pm\lambda)\nu/r_1, \; V_r\Big|_{r=r_2} = (\pm\lambda)\nu/r_2,$$

where λ is the magnitude characterizing the strength of injection/suction. For the hydrodynamic stream function, these boundary conditions read

$$R'(\bar{r}_1) = -r_2^2\omega_1\bar{r}_1/\nu = -\mathrm{Re}_1\bar{r}_1, \; R'(1) = -r_2^2\omega_2/\nu = -\mathrm{Re}_2 .$$

Here, Re_1 and Re_2 are the Reynolds numbers calculated by the angular velocities of the inner and outer cylinders, respectively, moreover, these numbers can be both positive and negative. A positive value of any Re means that a corresponding cylinder rotates anticlockwise, a negative value means that it rotates clockwise. The signs (+) or (–) in this situation denote the direction of cylinder rotation: (–) the cylinder rotates clockwise; (+) the cylinder rotates anticlockwise.

Since to define the velocity field it is necessary to derive a solution for the function $R'(r)$, it is enough to set only **one** additional condition. As such a condition, we can accept a condition of conservation of per second liquid flowmeter per ring unit length along the *z*-axis in the cross-sections $\varphi = \mathrm{const}$ because it does not depend on the liquid suction/injection:

$$Q = \int_{r_1}^{r_2} V_\varphi dr = \mathrm{const} = -\nu \int_{\bar{r}_1}^{1} R'(\bar{r}) d\bar{r} = -\nu \left[R(1) - R(\bar{r}_1)\right].$$

Introducing a mean flowmeter velocity $V_{\varphi\,\text{cp}} = \left(V_\varphi\big|_{r=r_1} + V_\varphi\big|_{r=r_2} \right) \big/ 2$, yields the required third condition

$$R(1) - R(r_1) = -\left(\text{Re}_1 r_1 + \text{Re}_2\right)\left(1 - r_1\right)/2 .$$

The problem has an analytical solution.

The results of calculations of the function $(-R'(r))$, which determines the azimuthal velocity, are presented in Figs.V.18 – V.20. Calculations were performed for cases, when the radius of the inner cylinder is twice less than the radius of the outer cylinder ($r_1 = 0.5$) and the cylinders rotate in opposite directions with equal angular velocities ($\text{Re}_1 = 10$, $\text{Re}_2 = -10$). If the angular velocity of, e.g., the inner cylinder changes in sign and absolute value, then, as numerical experiments show, the point of measure (count) of the velocity profile at $r = r_1 = 0.5$ shifts due to this change, not changing the general character of the profile behaviour at varying λ, Ha^2 and $\overline{\text{Ha}}^2$. The same can be referred to the variation of the angular velocity of the outer cylinder.

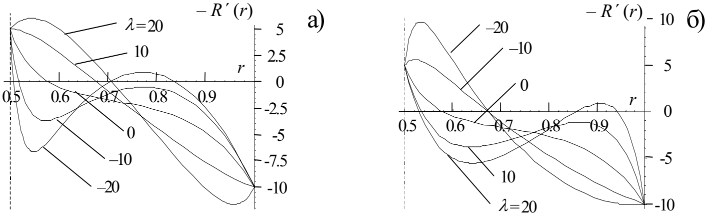

Fig.V.18. Azimuthal velocity profiles vs. different directions of the azimuthal magnetic field determined by the value of suction/injection, at $\text{Ha}^2 = 100$. a) $\overline{\text{Ha}}^2 = +100$; b) $\overline{\text{Ha}}^2 = -100$.

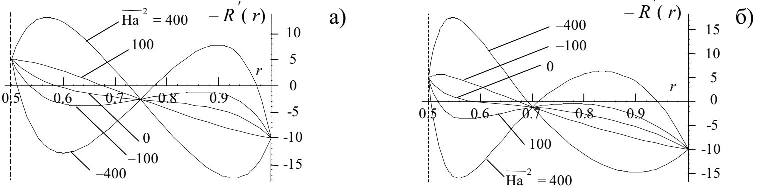

Fig.V.19. Azimuthal velocity profiles vs. the value and direction of the azimuthal magnetic field at a predetermined value of the radial field, at $\text{Ha}^2 = 100$.

From the definition of Ha^2 and $\overline{\text{Ha}}^2$ it follows that the radial magnetic field B_r direction is in principle insignificant because this field component is quadratic in the equation of motion. As to $\overline{\text{Ha}}^2$, then, due to the fact that it is defined by the product of B_r and B_φ, a corresponding term in the equation of motion affects the flow field due to the mutual orientation of these fields. The results presented in Figs.V.18 – V.20 are referred to the positive direction of the radial magnetic field – from the symmetry axis. In this case, the sign (–) in front of $\overline{\text{Ha}}^2$ denotes the clockwise direction of the azimuthal magnetic field, the sign (+) indicates anticlockwise direction. It follows from Fig.V.18 that in the absence of suction/injection ($\lambda = 0$) the radial magnetic field forms a S-shaped profile of the azimuthal velocity, typical of MHD Couette flow. Suction ($-\lambda$) of the liquid away from the inner cylinder with the direction of azimuthal magnetic field corresponding to $+\overline{\text{Ha}}^2$ sharply decreases the azimuthal velocity in the vicinity of this cylinder (Fig.V.18a), even to the point that the rotation of liquid changes its direction.

At the outer cylinder, over the surface of which the liquid is injected, an opposite process takes place at this time. Injection ($+\lambda$) through the inner cylinder first makes the velocity profile straight, as if minimizing the action of the radial magnetic field, and then, as λ grows, causes formation of zones at the both cylinders, where the angular velocity of rotation of liquid exceeds the angular velocities of the cylinders' rotation.

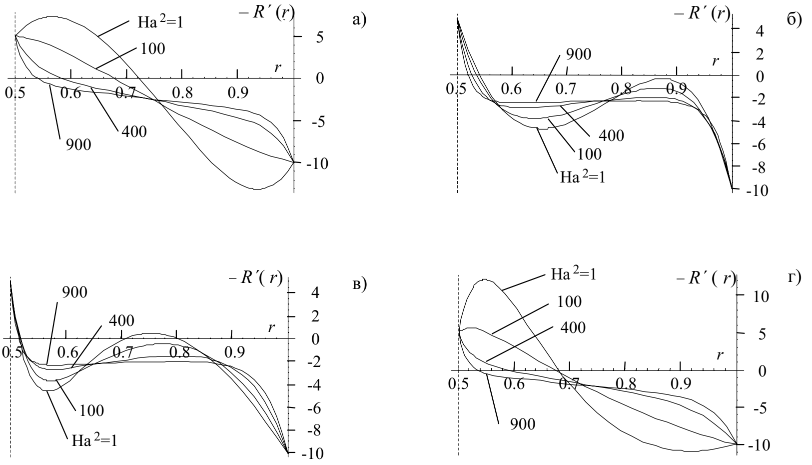

Fig.V.20. Influence of the radial magnetic field on the azimuthal velocity for different directions of the azimuthal magnetic field and suction/injection.

a) $\lambda = +10$, $\overline{\text{Ha}}^2 = +100$; b) $\lambda = +10$, $\overline{\text{Ha}}^2 = -100$;

c) $\lambda = -10$, $\overline{\text{Ha}}^2 = +100$; d) $\lambda = -10$, $\overline{\text{Ha}}^2 = +100$.

If change the direction of the azimuthal magnetic field ($-\overline{\text{Ha}}^2$), the above process of velocity profile deformation takes place in the reverse sequence (Fig.V.18b).

With the same predetermined value of the radial magnetic field ($\text{Ha}^2 = 100$) and injection of the liquid through the inner cylinder, the increase of $\overline{\text{Ha}}^2$ value at (+) provides almost the same result (Fig.V.19a) like the increase of the injection value λ at the predetermined $\overline{\text{Ha}}^2$ (Fig.V.18a), and the sign (–) provides a result similar to the increase of suction value ($-\lambda$). If the liquid is suctioned through the inner cylinder (Fig.V.19b), the influence of the value and sign of the parameter $\overline{\text{Ha}}^2$ is opposite to injection. Similar to the situation illustrated in Fig.V.18, liquid zones are formed with typical much higher angular velocities of the cylinders' rotation at injection $\overline{\text{Ha}}^2 > 0$, or zones with the reverse rotation of the liquid at $\overline{\text{Ha}}^2 < 0$. Suction of the liquid changes the result to opposite.

Finally, the increase of the radial magnetic field at the predetermined parameters of suction/injection λ and $\pm\overline{\text{Ha}}^2$ minimizes their influence: at $\text{Ha}^2 \gg 1$ a flow core is formed, which is almost alike for the parameter values chosen as illustrations in Fig.V.20; the velocity profiles become S-shaped, typical of the MHD Couette flow, with the formation of pronounced boundary layers near to the rotating surfaces.

The influence of radial and azimuthal magnetic fields and suction/injection on the flow has been analyzed. Trajectories of particle motion in the liquid have been obtained at different combinations of suction/injection and magnetic fields.

The friction of the rotating walls is

$$\tau_{\mathrm{w}} = \mu\, \partial V_{\varphi} \big/ \partial r \,\Big|_{r_1, r_2} = -\mu\, k\, R'' \big/ r_2^2 = \mu\, \omega_1\, R'' .$$

In general, the values of τ_{w} on the inner $r = r_1$ and on the outer $r = 1$ cylinders can be obtained by differentiating the derived solutions. Let us analyze the influence of each parameter Ha, $\overline{\mathrm{Ha}}$ and λ increase on the friction.

With the limit at $\mathrm{Ha} \to \infty$, we get the values of R'' on the rotating walls correct to the order of terms $\overline{\mathrm{Ha}}^2 / \mathrm{Ha}^2$:

$$R''(r_1) = -\frac{r_1\,\mathrm{Re}_1 + \mathrm{Re}_2}{1 + r_1} + \frac{\mathrm{Ha}(\mathrm{Re}_1 - \mathrm{Re}_2)}{1 + r_1} - \frac{\overline{\mathrm{Ha}}^2 \lambda \ln r_1}{\mathrm{Ha}^2} + \frac{2\overline{\mathrm{Ha}}^2 \lambda \ln r_1}{\mathrm{Ha}(r_1^2 - 1)} - \frac{2(\mathrm{Re}_1 - \mathrm{Re}_2)}{\mathrm{Ha}(1 + r_1)^2},$$

$$R''(1) = \frac{\mathrm{Ha}\, r_1(\mathrm{Re}_1 - \mathrm{Re}_2) - (r_1\,\mathrm{Re}_1 + \mathrm{Re}_2)}{1 + r_1} + \frac{\overline{\mathrm{Ha}}^2 \lambda (1 - \ln r_1)}{\mathrm{Ha}^2} - \frac{2\overline{\mathrm{Ha}}^2 \lambda \ln r_1}{\mathrm{Ha}(r_1^2 - 1)} - \frac{2(\mathrm{Re}_1 - \mathrm{Re}_2)}{\mathrm{Ha}(1 + r_1)^2} + \frac{\overline{\mathrm{Ha}}^2 \lambda}{\mathrm{Ha}} .$$

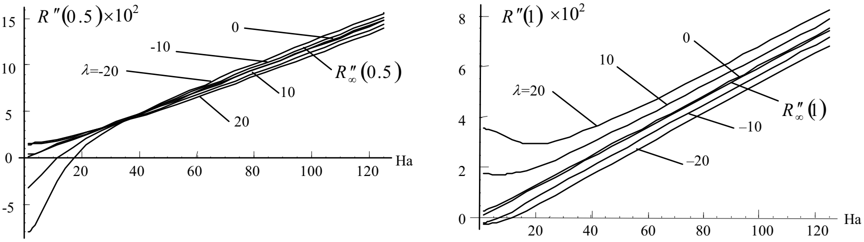

Fig.V.21. Friction on the inner (left-hand side) and on the outer (right-hand side) cylinders vs. Ha at different λ. $\overline{\mathrm{Ha}}^{\,2}=+10$.

The terms of order $\overline{\mathrm{Ha}}^2 / \mathrm{Ha}^2$ must be kept because the increase of $\mathrm{Ha}^2 \sim B_r^2$ means the simultaneous increase of $\overline{\mathrm{Ha}}^2 \sim B_r\, B_\varphi$, therefore, these terms decrease only at predetermined λ and B_φ with a growing B_r and only like B_r^{-1}. This explains the situation in Fig.V.21, where curves $R''(0.5)$ and $R''(1)$ plotted for moderate Ha values clearly show the influence of suction/injection λ on the friction on the rotating cylindrical surfaces. With predetermined λ and B_φ and at $\mathrm{Ha} \to \infty$, a strong radial magnetic field completely minimizes the azimuthal magnetic field and parameter λ presence. This also agrees with the behaviour of the velocity field at $\mathrm{Ha} \to \infty$ (Fig.V.20). As to the parameter $\overline{\mathrm{Ha}}^2$ influence, it should be said that, firstly, such influence manifests itself only in combination with $\overline{\mathrm{Ha}}^2(\pm\lambda)$ (see the equation of motion) and, secondly, at predetermined values of B_r the friction varies linearly with the growth of $|B_\varphi|$, and, besides, such variation is determined by the mutual orientation of λ, B_φ and B_r.

Conclusions

The goal of the current work is to generalize and methodically arrange the methods allowing to derive self-similar solutions for hydrodynamic and magnetohydrodynamic problems in the precise formulation and in the boundary layer approximation.

It has been shown by the thesis author and by other authors that the application of the self-similar approach yields similar equations, which differ only in coefficients. It means that a universal equation, or a set of equations can be derived, which describe all possible self-similar solutions in a given system of coordinates, where the coefficients are determined by the conditions of a definite problem under solution.

The results of the performed investigations allow the following conclusions.

I. Hydrodynamics.

Precise Formulation:

1. A universal self-similar equation has been derived in the Cartesian coordinates, which describes all possible in such formulation plane problems; an exact equation is derived describing plane problems, which can be solved by the method of separation of variables that, as the author knows, has been done for the first time.
2. It has been shown that it is impossible to transform the Navier-Stokes equations to a self-similar ones in the polar coordinates; the method of variable separation allowed to derive some versions of the equations determined by the variables r and φ, and their possible solutions have been analyzed.
3. Three possible versions of self-similar solutions and two solution versions have been derived by the method of separation of variables in the cylindrical coordinates in the precise axisymmetric formulation.
4. The impossibility to transform the Navier-Stokes equations to their self-similar form in the axisymmetric formulation in the spherical coordinates has been proved; it is shown that by applying the method of variable separation, the axisymmetric equations in precise formulation can be transformed only to equations relative to variable θ.

Boundary Layer Approximation:

1. In the polar and spherical (axisymmetric formulation) coordinates it has been found impossible to transform the Navier-Stokes equations to boundary layer equations; the only way to derive such equations is to tend the parameters, characterizing the problem, towards some critical values.
2. In the Cartesian coordinates, a universal self-similar equation has been derived, which describes all possible in this formulation plane flows in the boundary layer approximation; it has been shown that there is no ground to explicitly apply the method of variable separation, but there is a version at which the derived equation corresponds to the method of variable separation, and, besides, this equation is not only a boundary layer equation but also an exact equation, too.
3. In the cylindrical coordinates, 11 versions of problem derivation have been obtained in an axisymmetric formulation in the boundary layer approximation (with account for azimuthal rotation); 2 versions of the above eleven had not been considered previously and have no precise physical interpretation (as the author knows, the problem has not been analyzed in detail previously); as to the Cartesian coordinates, there is no ground to apply the method of variable separation, but five of the above eleven versions are, in fact, the separation of variables and exact equations as well.

II. Magnetohydrodynamics

Precise Formulation:

1. For plane flows in the Cartesian coordinates it has been shown that the application of the self-similar approach makes possible to describe only those problems, where the magnetic field does not affect the velocity field; by applying the method of variable separation it is possible to solve problems, where the velocity field interacts with the magnetic field (this version has not been considered anywhere before).
2. In the polar coordinates, it is possible to derive equations in precise formulation only relative to the variable r.
3. In the cylindrical coordinates, in an axisymmetric case, it is possible to derive five versions of the solution, three of which correspond to the method of separation of

variables.

4. In the spherical coordinates, in an axisymmetric case, it is possible to derive only the equations relative to the variable θ.
5. The non-inductive approximation for all four systems of coordinates has been considered.
6. It is possible to derive a magnetohydrodynamic solution in the non-inductive approximation in the Cartesian coordinates only by the method of separation of variables, not by the self-similar method, that has been done by the author. It has been shown that the only magnetic field, which affects the velocity field, is a magnetic field with a neutral point.
7. In the polar coordinates in the non-inductive approximation, a class of applicable external magnetic fields has been revealed; the author has analyzed some versions of the applicable magnetic fields and presented schemes of force line distributions for them; also, universal equations relative both to r and φ variables have been derived; a version has been found allowing to transform the Navier-Stokes-Maxwell equations into ordinary differential equations not by separating the variables, but by choosing another stream function related to the problem geometry.
8. In the non-inductive approximation in the cylindrical coordinates for an axisymmetric case possible types of external magnetic fields have been found, which allow to transform the equations with partial derivatives into ordinary differential equations in all five versions; some fields have been interpreted from the physical point of view.
9. In the non-inductive approximation, in the spherical coordinates for an axisymmetric case, applicable distributions of electric and external magnetic fields have been obtained.

Boundary Layer Approximation:

1. The equations only in the Cartesian and cylindrical coordinates have been considered because they agree with the equations in precise formulation in both the polar and the spherical coordinates; the precise formulation free of any assumptions for electromagnetic forces and the non-inductive approximation have been considered, too.
2. In the Cartesian coordinates in the precise boundary layer approximation formulation, two versions of self-similar equations have been derived, one of which is universal; applicable magnetic fields in the non-inductive approximation have been defined and physically interpreted. The author has found a version, which allows to derive an equation, which simultaneously is both an exact and an equation in the boundary layer approximation; this version corresponds to the method of variable separation, and the only applicable magnetic field is the magnetic field with a neutral point.
3. In the cylindrical coordinates for an axisymmetric case, 9 possible versions of the boundary layer solution with account for the azimuthal rotation in precise formulation have been considered; as a result, it has been found that only in 7 of the above versions the magnetic fields affects the velocity field; acceptable distributions of electric current and magnetic field have been determined for all cases.
4. In the non-inductive approximation in the cylindrical coordinates, also nine possible versions of self-similar solutions in the boundary layer approximation have been discussed; in this case, all nine versions describe the MHD flows; applicable magnetic field and their physical interpretations are analyzed for all nine versions.

For all the above discussed versions, equations or sets of equations have been derived, which can be used for solving definite problems without additional transformations. For this purpose, it is enough to choose a stream function with reference to the geometry of the flow zone and other problem conditions. All necessary parameters in the equations are derived by simple substitution of coefficients, entering into the stream functions. In other words, the obtained results can be used as some reference book, allowing to avoid calculations of the same type. No complete classification has been provided earlier. Previously, detailed, yet, less complete systems had been proposed in monographs [43], [70] and [72], but the first two monographs restricted their consideration to the spherical and cylindrical coordinate systems.

Section V deals with the problems, which have been resolved in the framework of the proposed approach. The solution of some known problems allows to compare the proposed approach with the previously applied methods and estimate its efficiency.

Some new results of the problems' solution discussed in Section V are the following.

1. The problem of MHD flow in diffuser/confuser in an azimuthal field determines the dependence of friction on the wall on the angle of diffuser/confuser opening, and for the diffuser flow a new dependence between the Hartmann number, which characterizes the magnetic field, and the critical Reynolds number, at which the flow breaks off from the diffuser wall, has been found. This dependence differs from the one considered in [67] in a way that there is a dependence on the angle of confuser opening and that the critical Reynolds number quadratically, not linearly as in [67], depends on the Hartmann number.
2. The problem of MHD flow with suction/injection of a plate in the azimuthal magnetic field investigates a possibility of flow break-off at the varying magnetic field induction and suction/injection intensity. It has been found that for the confuser flow at any combinations of the magnetic field and suction/injection, no break-off takes place, but a reverse flow is driven. For the diffuser flow, dependencies of the critical Reynolds number, at which the flow breaks off, on the Hartmann number and suction/injection intensity have been found.
3. The problem of MHD flow with suction/injection on a plate in a radial magnetic field also investigates the possibility of flow break-off. In this case, the break-off is possible in both diffuser and confuser flows that is determined by a flow induced by the rotor of electromagnetic force. Ranges of the Hartmann number values and suction/injection intensities have been defined, at which the flow does not break off. Two zones of no-breaking off flow have been found, one of which is determined by large enough intensities of suction/injection.
4. The study of the problem of circular MHD analogue of the Couette flow has provided exact solutions as for the radial magnetic field as for pure hydrodynamics. Exact solutions were obtained and the friction of the walls was calculated. The asymptotics of friction has been found at the increase of the Hartmann number towards infinity. The formation of a boundary layer has been shown as with the increase of the Hartmann number, i.e., when the radial magnetic field becomes stronger, as at variations of the cylinders' rotation rate.
5. The study of the circular MHD flow between two rotating cylinders with suction/injection has provided an exact solution for the velocity field. The influence of suction/injection strength and radial and azimuthal magnetic fields at a constant ratio of rotational velocities on the flow was investigated. The interaction of external magnetic fields and suction/injection was analyzed; the trajectories of particle motion in the liquid were plotted, and the friction on the rotation walls was calculated. It occur that due to the presence of two independent magnetic field components, two Hartmann numbers can be considered, one of which depends only on the radial magnetic field value, the other is caused by a combination of radial and magnetic fields. The variation of these magnitudes significantly affects the flow velocity profile and drives a reverse flow, too.

REFERENCES

1. Aristov, S.N. and Gitman, I.M., *Viscous flow between two moving parallell disks: exact solutions and stability analysis*, J. Fluid Mech., Vol. 464, pp. 209-215, 2002.
2. Barenblatt, G.I. and Zel'dovich, Ya.B., *Self-similar solutions as intermediate asymptotics*, Annual Rev. of Fluid Mech., Vol. 4, pp. 285-312, 1972.
3. Berman, A.S., *Laminar flow in channels with porous walls*, J. Appl. Physics, Vol. 24, No. 9, pp. 1232-1235, 1953.
4. Blasius H., *Crenzshichten in Flussigkeiten mit kleiner Reibung*, Zeitschr. f. Math. u. Phys., bd. 56, s. 1-37, 1908.
5. Bluman, G.W. and Cole, J.D., *Similarity Methods for Differential equations*, Springer-Verlag, New York, 1974.
6. Bödewadt V.T., *Die Drehströmung über festen Gründe*, Z. Angew. Math. Mech. – 1940, 20, H.5. – S. 241 – 253.
7. Burde, G.I., *The construction of special explicit solutions of the boundary –layer equations. Steady flows*, Quart. J. Mech. Appl. math., Vol. 47, No. 2, pp. 247-260,1994.
8. Dresner, L., *Similarity Solutions of Nonlinear Partial Differential Equations*, Pitman, Boston, 1983.
9. Falkner, V.M. and Skan, S.W., *Some approximate solutions of the boundary layer equations*, Phil. Mag., Vol. 12, pp.865-896, 1931.
10. Goertler H., *Berechnung von Ausgaben der freien Turbulenz auf Grund neuen Naherungsansatzes*, ZAMM, Vol. 22, No. 5, pp. 244-254, 1942.
11. Hamell G., *Spilalformige Bewegung zaher Flussig keiten*, d. Dt. Mathematiker – Vereinigung, Vol. 25. s. 34-60, 1916.
12. Hartman J. *Hg–dynamics, I. Theory of the laminar flow of an electrically conductive liquid in a homogeneous magnetic field*, Mat.–fys. medd. K. Danske videnskab. – 1937. –Bd. 15, № 7.
13. Jeffery, G.B., *The two-dimensional steady motion of a viscous fluid*, Phil. Mag. Ser. 6, vol. 9, pp. 455-465, 1915.
14. Karman Th., *Uber laminare und turbulent Reibung*, ZAMM, Bd. 1, pp. 233-251, 1921.
15. Kremenetsky V., *About the Self-similar Heat Boundary Layer Problems In Magnetohydrodynamics,* MATHEMATICAL MODELLING AND ANALYSIS, Vol.7, 2002.
16. Kremenetsky V., *MHD analogues of the Couette flow with suction/injection,* Proceedings of the 5th International PAMIR Conference "Fundamental and Applied MHD", Vol. I, Ramatuelle, France, September 16-20, 2002.
17. Kremenetsky V., *On the Self-similar Heat Boundary Layer Problems In Magnetohydrodynamics,* Magnetohydrodynamics, Vol. 37, No. 4, p. 373-378, 2001.
18. Kremenetsky V., Shcherbinin E., *Boundary layers formation in case of free and near-the-wall fan jets at the electrically induced vortical flows*, Mahyd'95. Abstracts. p. 27. Jurmala, Latvia,1995.
19. V. Kremenetsky, E. Shcherbinin *On some magnetohydrodynamic flows in polar coordinates*, Proceedings of the 5th International Pamir Conference on Fundamental and Applied MHD, Ramatuelle, France, 2002, Vol. 1, VII-7.
20. Kremenetsky V., Shilova E., Shcherbinin E., *On Magnetohydrodynamic Flows in Convergent and Divergent Diffuser*, Magnetohydrodynamics, Vol. 39, No. 2, p. 179-186, 2003.
21. Kremeņeckis V., Ščerbiņins E., *Robežslāņa teorija hidrodinamikā aksiāli simetriskām un virpuļu plūsmām*, 35. RTU zinātniskā un tehniskā konference 1994. g. 19.-22. aprīlī. Konferences materiāli. lpp. 34-37. Rīga, 1994.

22. Lin C.C., *Note on a class of exact solutions in magnetohydrodynamics*, Arch. Rational Mech. Anal., Vol. 1, No. 4, pp. 391-395, 1958.
23. Ludlow, D.K., Clarkson, P.A., and Bassom, A.P., *New similarity solutions of the unsteady incompressible bounadry-layer equations*, Quart. J. Mech. and Appl. Math., Vol. 53, 175-206, 2000.
24. Ma, P.K.H. and Hui, W.H., *Similarity solutions of the two-dimensional unsteady boundary-layer equations*, J. Fluid. Mech., Vol. 216, pp. 538-540, 1990.
25. Meleshko, S.V. and Pukhnachov, V.V., *A class of partially invariant solutions of Navier-Stokes equations*, J. Appl. Mech. & Tech. Phys., Vol. 40, No. 2, pp. 24-33, 1999.
26. Pohlhausen K., *Zur naherungsweisen Integration der Differentialgleichung der laminaren Grenzschicht*, ZAMM, Vol. 1, s. 252-268, 1921.
27. Polyanin A.D., *Exact solutions to the Navier-Stokes equations with generalized separation of variables*, Doklady Physics, Vol. 46, No. 10, pp. 726-731, 2001.
28. Prandtl L., *Uber Flussgkeitsbewegung bei sehr kleiner Reibung*, Verhanlung d. III Internat. Mathem. Kongr. Heidelberg, s. 484-491, 1904.
29. Ruediger G., *MHD Instabilities in Hydromagnetic Taylor-Couette Flows*, Proceedings of The 15th Riga and 6th PAMIR Conference on Fundamental and Applied MHD, Vol. 1, pp. 117-120, June 27 - July 1, 2005, Rigas Jurmala, Latvia.
30. Shcherbinin E., Bartulis A., Kremenetsky V., *Theory of axisymmetric boundary layer in hydrodynamics and magnethydrodynamics*, 3rd International Conference on Transfer Phenomena in Magnetohydrodynamics and Electroconducting Flows, vol. 1, p. 255-260, Aussois, France, 1997.
31. Shcherbinin E., *MHD flows in polar coordinates*, Magnetohydrodynamics, Vol. 39, No. 2, 169-178, 2003.
32. Squire H.B., *Jet emerging from a hole in a plane wall*, Phil. Mag., Ser. 7, vol. 43, No. 343, pp. 942-945, 1952.
33. Squire H.B., *The round laminar jet*, Quart. J. Mech. and Appl. Math., Vol. 4, No. 3, pp. 321-329.
34. Wang, C.Y., *Exact solutions for the Navier-Stokes equations – the generalizd Beltrami flows, review and extensions*, Acta Mech., Vol. 81, pp. 69-74, 1990.
35. Yuan S.W., Finkelstein A.B., Brooklyn N.Y., *Laminar pipe flow with injection and suction through a porous wall*, Trans. ASME, Vol. 78, No. 4, pp. 719-724, 1956.
36. Акатнов Н.И., *Распространение плоской ламинарной струи несжимаемой жидкости вдоль твердой стенки*, Труды Ленингр. Политех. ин-та, Техническая гидромеханика, вып. 5, с. 24-31б 1953.
37. Бай Ши-И, *Теория струй*, М., 1960.
38. Бартулис А., Кременецкий В., Щербинин Э. В., *Общий метод построения автомодельных решений для плоских МГД течений,* Магнитная гидродинамика, Vol. 34, N2, c.109-128, 1998.
39. Бартулис А., Кременецкий В., Щербинин Э., *Теория осесимметричного пограничного слоя II рода*, Магнитная гидродинамика , Vol. 32, N1, c.291-299, 1996.
40. Бартулис А., Щербинин Э. В., *Магнитоуправляемое отсасывание пограничного слоя*, Магнитная гидродинамика, Vol. 34, No. 3, 211-224, 1998.
41. Бартулис А., Щербинин Э.В., *О методике численного интегрирования уравнений автомодельного пограничного слоя*, Магнитная гидродинамика, Vol. 36, No. 2, 119-152, 2000;
42. Бартулис А., Щербинин Э.В., *Электровихревое течение между двумя вращающимися дисками*, Магнитная гидродинамика, vol.33, No.2, 156-162, 1997.
43. Бояревич В.В., Фрейберг Я.Ж., Шилова Е.И., Щербинин Э.В., *Электровихревые течения*, «Зинатне», 1985.
44. Бэтчелор Дж.К., *Введение в динамику жидкости*, «Мир», Москва, 1973.

45. Волосевич П.П., Леванов Е.И., *Автомодельные решения задач газовой динамики и теплопереноса*, Изд-во МФТИ, М., 1997.
46. Вулис Л.А., Джаугаштин К.Е., *Магнитогидродинамическое течение Куэтта*, ЖТФ, Т. 34, №. 12, с. 2171-2177, 1964.
47. Гельфгат Ю.М., Лиелаусис О.А., Щербинин Э.В., *Жидкий металл под действием электромагнитных сил*, «Зинатне», Рига, 1976.
48. Гольдштик М.А., *О закрученных струях*, Механика жидкости и газа, №1, 1979.
49. Гольдштик М.А., *Один класс точных решений уравнений Навье-Стокса*, ПМТФ, №2, 1966.
50. Гольдштик М.А., Штерн В.Н., Яворский Н.И., *Вязкие течения с парадоксальными свойствами*, «Наука», Сиб. отделение, Новосибирск, 1989.
51. Гринспен Х., *Теория вращающихся жидкостей*, Гидрометеоиздат, Ленинград, 1975.
52. Кочин Н.Е., Кибель И.А., Розе Н.В., *Теоретическая гидромеханика, ч. 1*, Физматгиз, М., 1963.
53. Кочин Н.Е., Кибель И.А., Розе Н.В., *Теоретическая гидромеханика, ч. 2*, Физматгиз, М., 1963.
54. Ладыженская О.А., *Математические вопросы динамики вязкой несжимаемой жидкости*, «Наука», М., 1970.
55. Ландау Л.Д., Лифшиц Е.М., *Теоретическая физика. Т. 6. Гидродинамика*, «Наука», М., 1988.
56. Ландау Л.Д., *Новое точное решение уравнения Навье-Стокса*, ДАН, т. 43, № 7, с. 299-301, 1944.
57. Лойцянский Л.Г., *Ламинарный пограничный слой*, «Наука», Москва, 1962.
58. Лойцянский Л.Г., *Механика жидкости и газа*, «Наука», Москва, 1978.
59. Лойцянский Л.Г., *Радиально-щелевая струя в пространстве, заполненном той же жидкостью*, Тр. Ленингр. политех. инст., № 5, т. 17, вып. 1, с. 3-16, 1953.
60. Мартыненко О.Г., Коровкин В.Н., Соковишин Ю.А., *Теория ламинарных вязких струй*, «Наука и техника», Минск, 1985.
61. Седов Л.И., *Методы подобия и размерностей в механике*, «Наука», М., 1988.
62. Седов Л.И., *Механика сплошной среды, т. 1*, «Наука», М., 1970.
63. Седов Л.И., *Механика сплошной среды, т. 2*, «Наука», М., 1970.
64. Слезкин Н.А., *Движение вязкой жидкости между двумя конусами*, Уч. зап. МГУ, № 2, с. 83-87, 1934.
65. Смирнов Е.М., *Автомодельное решение уравнений Навье-Стокса для закрученного течения жидкости в круглой трубе*, ПММ, т. 45, вып. 5, с. 833-839.
66. Сычев В.В., *О движении вязкой электропроводной жидкости под действием вращающегося диска в присутствии магнитного поля*, ПММ, 1960, т. 24, вып. 5, с. 906-908, 1960.
67. Цинобер А.Б., *Магнитогидродинамическое обтекание тел*, «Зинатне», Рига, 1970.
68. Шлихтинг Г., *Теория пограничного слоя*, М., 1974.
69. Щербинин Э.В., *Магнито-вязкий пограничный слой,* Магнитная гидродинамика, Vol. 35, No. 3, 217-235, 1999.
70. Щербинин Э.В., *Струйные течения вязкой жидкости в магнитном поле*, «Зинатне», Рига, 1973.
71. Щербинин Э.В., *Теория осесимметричного пограничного слоя в гидродинамике и магнитной гидродинамике*, Магнитная гидродинамика, Vol. 31, No. 2, 151-168, 1995.

72. Щербинин Э.В., *Теория пограничного слоя в гидродинамике и магнитной гидродинамике, т. 1. Плоские течения*, Лаборатория ЭВТ, самиздат, Саласпилс, 2004.
73. Яцеев В.И., *Об одном классе точных решений уравнений движения вязкой жидкости*, ЖТФ, т. 20, № 11, с. 1031-1034, 1950.